Coursework
exam3 3/27–3/29
hw07 TBA, probably due 4/7
Course Map
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- **Computational Thinking**: types, operators, expressions
- **Data Processing**: lists, dictionaries, loops
- **Numerical Simulation**: arrays, plotting, RNGs
- **Problem Solving**: equation solving, optimization ★
- **Raw Data & Modeling**: MATLAB, statistics, curve fitting
Equations on Computers
How do we represent equations on computers?
We often just compose a function and write some expressions, or maybe have a series.
In other words, we are concerned with float representations—we want numbers out of them.
Many times we represent the function as a pair of arrays, x and y (like for plotting).
We can do so symbolically as well (see HPL on SymPy), but won’t in 101.
Suppose you wish to evaluate the function:

\[ y = a \sin^3 x + b \sin^2 x + c \sin x + d \]
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On a computer, which way is better?

\[ y = a*\sin(x)**3 + b*\sin(x)**2 + c*\sin(x) + d \]

\[ t = \sin(x) \]

\[ y = a*t**3 + b*t**2 + c*t + d \]
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\[ t = \sin(x) \]
\[ y = a*t^{**3} + b*t^{**2} + c*t + d \]

The first way takes three times longer!
What about calculating $\pi$ using the Monte Carlo method?
Let's say, versus a series solution?
import numpy.random as npr

def mc_pi( n ):
    xy = npr.rand( n,2 ) * 2 - 1
    n_circle = 0
    for pair in xy:
        if ( pair[0]**2 + pair[1]**2 )**0.5 < 1.0:
            n_circle += 1
    estimate = n_circle / n * 4.0
    return estimate
def series_pi( n ):
    result = 0
    for k in range( 1, n ):
        term = ( (-1) ** (k+1) ) / (2 * k - 1)
        result += term
    return result*4
Which way is more efficient computationally?
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The series solution is much better, and other better ways may exist.

Later we’ll learn how to quantify this (probably lec19).
How do you calculate the value of \( \sin x \) or \( \exp x \)?
How do you calculate the value of $\sin x$ or $\exp x$?

$$\exp(-x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + ...$$

$$\exp(-x) = \frac{x^0}{0!} - \frac{x^1}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$
How do you calculate the value of \( \sin x \) or \( \exp x \)?

\[
\exp(-x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots
\]

\[
= \frac{x^0}{0!} - \frac{x^1}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\]

This series is well-behaved, but...
Intermediate terms can behave like:

\[
\frac{10^5}{5!} = \frac{100,000}{120} = 833.333
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\[ \frac{10^5}{5!} = \frac{100,000}{120} = 833.333 \]

or

\[ \frac{10^{12}}{12!} = \frac{1,000,000,000,000}{479,001,600} = 2,087.675 \]

Very large numbers result, leading to inefficient calculation and possible numerical error.

So what can we do?
In this case, the solution is to exploit a property of $e$, namely

$$e^x = \frac{1}{e^{-x}}$$

and thus use quantities less than zero for $x$. 
What about trying to find a “better” way to solve a problem?
Example: collinearity of three points
If you find this sort of analysis interesting, I recommend:

- CS 450, TAM 470
- Forman Acton’s marvelous *Numerical Methods that Work* (1970)
- Abramowitz & Stegun, *Handbook of Mathematical Functions* (1964)
Suppose that you wish to evaluate the function:

\[ t(x) = a \exp(3x) + b \exp(2x) + c \exp(x). \]

On a computer, which is better?

A. \[ t = a*\exp(3*x) + b*\sin(2*x) + c*\sin(x) \]

B. \[ z = \exp(x) \]
\[ t = a*z**3 + b*z**2 + c*z + d \]
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\[ t(x) = a \exp(3x) + b \exp(2x) + c \exp(x). \]

On a computer, which is better?

A \( t = a*\exp(3*x) + b*\sin(2*x) + c*\sin(x) \)

B \( z = \exp(x) \quad t = a*z**3 + b*z**2 + c*z + d \)

★
Solving Equations in x
Next, let’s consider how to find a specific solution to an equation, a value of $x$ for which $f(x)$ has a desired property.
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The easiest way is to plot LHS v. RHS and find the crossover point:
$x^2 + 5x - (2x^2 - 3) = -2x^2 - x$

$x**2 + 5*x - (2*x**2 - 3) == -2*x**2 - x$
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\[ x^{**2} + 5*x - (2*x^{**2} - 3) == -2*x^{**2} - x \]

\[ x = \text{np.linspace}( -10,10,1001 ) \]

\[ \text{lhs} = x^{**2} + 5*x - (2*x^{**2} - 3) \]

\[ \text{rhs} = -2*x^{**2} - x \]

\[ \text{plt.plot}( x,\text{lhs},'r' , x,\text{rhs},'b' ) \]

\[ \text{plt.plot}( x,\text{lhs-rhs},'g' ) \]
\[ x^2 + 5x - (2x^2 - 3) = -2x^2 - x \]

\[ x^{**2} + 5x - (2*x^{**2} - 3) == -2*x^{**2} - x \]

\[ x = \text{np.linspace}(-10,10,1001) \]

\[ \text{lhs} = x^{**2} + 5x - (2*x^{**2} - 3) \]

\[ \text{rhs} = -2*x^{**2} - x \]

\[ \text{plt.plot}( x,\text{lhs},'r', x,\text{rhs},'b' ) \]

\[ \text{plt.plot}( x,\text{lhs-rhs},'g' ) \]

❖ This works, but we need something better than eyeballing it.
Newton’s method uses the function and its derivative to locate the $x$ of the zero, $x^*$. The trick, of course, is that you need the derivative.
def dfdx( f,x,h=1e-3 ):
    return ( f( x+h ) - f( x ) ) / h

def newton( f,x0,tol=1e-3 ):
    d = abs( 0 - f( x0 ) )
    while d > tol:
        x0 = x0 - f( x0 ) / dfdx( f,x0 )
        d = abs( 0 - f( x0 ) )
    return ( x0,f( x0 ) )
Equations

\[ \cos x + 2 = x^3 - x^2 \]

def eqn( x ):
    return ( np.cos( x ) + 2 ) - ( x**3 - x**2 )
The preceding code works okay, but a full implementation is available as `scipy.optimize.newton(f, x0)`.
We can also find minima using `scipy.optimize.fmin( f, x0 )`. 
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This requires you to be clever in preparing f: you may have to negate or manipulate your function.
On vacation, you purchase a range of $n$ souvenirs of varying weight and value. When it comes time to pack, you find that your bag has a weight limit of 50 pounds. What is the best set of items to take on the flight?
Next steps
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- exam3 3/27–3/29
- hw07 TBA, probably due 4/7
- No quiz this time—study for exam3 instead!
- Read for the next class