newton's method and optimization

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review

Friday December 11 – Tuesday December 15 Final Exam
(computerized facility)

1. Take a matrix

$$A = \begin{bmatrix} 0 & 5.2 & 0 \\ 15.5 & 0 & 3.88 \\ 9.1 & 4.0 & 0 \end{bmatrix},$$

   1.1 What is the CSR form?
   1.2 How many bytes does it take to store this CSR?
   1.3 What is the COO form?
   1.4 How many bytes does it take to store this COO?
   1.5 How much is a matrix-vector product with the matrix is dense?
   1.6 How much is a matrix-vector product with the matrix is sparse?
   1.7 What is a sparse matrix?
   1.8 Given a graph, form a sparse matrix.

2. Given three points, say $(-1, 0), (0, 1), (1, 0)$, what is the least-squares fit with a linear: $a_0 + a_1 x$?

   2.1 Form the least-squares problem.
   2.2 Find the coefficients using the normal equations.
   2.3 Find the coefficients using a QR factorization.
   2.4 What is a QR factorization?
   2.5 What is the residual in the least-squares problem? Is it zero?
   2.6 Plot your solution.

3. Given three points, say $(-1, 0), (0, 1), (1, 0)$, what is the least-squares fit with a quadratic: $a_0 + a_1 x + a_2 x^2$?

   3.1 Form the least-squares problem.

   3.2 Find the coefficients using the normal equations.

   3.3 What is the residual in the least-squares problem? Is it zero?

   3.4 Plot your solution.

4. Interpolate three points, say $(-1, 0), (0, 1), (1, 0)$, using a monomial basis: 1, x, x^2

   4.1 Form the Vandermonde matrix and right-hand side

   4.2 Solve the problem.

   4.3 Plot your solution.

   4.4 What is the difference between Least-squares and interpolation?

5. Interpolate three points, say $(-1, 0), (0, 1), (1, 0)$, using a Lagrange basis

    5.1 What is a Lagrange basis?

    5.2 What are the coefficients for the basis?

    5.3 Plot your solution.

    5.4 Does the interpolant depend on the basis? Why or why not?

6. Construct a *nonlinear* least-squares problem for fitting

$$f(x) = \alpha e^{\beta x}$$

to $(-1, 0), (0, 1), (1, 0)$

6.1 Can you think of other nonlinear least-squares problems?
6.2 Set up the problem.
6.3 What do you need to minimize?
6.4 Find a solution using Newton's method.
6.5 What are the coefficients for the basis?
6.6 Plot your solution.

7. Run one step of Newton's method to find (approximate) the square root of 612:

$$x^2 = 612$$

   7.1 What it's Newton's method to find a root?
   7.2 Set up the problem.
   7.3 Run one step with initial guess of 10.0.
   7.4 How fast does it converge?
   7.5 Does Newton's method always converge? Give an example or reason.
   7.6 How fast is Newton's method? What does this depend on?

8. Run one step of Newton's method to find (approximate) a spot where the following is locally minimized:

$$g(x) = x^3 + 4x^2 + 2x + 2$$

8.1 What it's Newton's method to find a minimum?

8.2 Set up the problem.

8.3 Run one step with initial guess of 1.0.

8.4 What is a condition for a minimum?

9. Suppose we have two nonlinear equations:

$$g_1(x, y) = x^2 + y^2 - 1 = 0$$
$$g_2(x, y) = x^4 - y^4 + xy = 0$$

Find a solution using Newton's method.

9.1 What is Newton's method to find the solution?
9.2 Set up the problem.
9.3 What is the Jacobian?
9.4 Run one step with initial guess of say $x = [1, 1]$

10. Suppose we have a 2D objective function:

$$p(x, y) = (x - y)^4 + 2x^2 + y^2 - x + 2y$$

Find a minimum to this function using Newton's method.

10.1 What is Newton's method to find the solution?

10.2 Set up the problem.

10.3 What is the gradient?

10.4 What is the Hessian?

10.5 Run one step with initial guess of say $x = [1, 1]$