# #5

## Norms

L. Olson

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Department of Computer Science
University of Illinois at Urbana-Champaign

# objectives

- Set up an array of data and measure its “size”
- Construct a “norm” and apply its properties to a problem
- Describe a “matrix norm” or “operator norm”
- Find examples where a matrix norm is appropriate and not appropriate

# vector addition and subtraction

Addition and subtraction are element-by-element operations

$$c = a + b \quad \Longleftrightarrow \quad c_i = a_i + b_i \quad i = 1, \ldots, n$$
$$d = a - b \quad \Longleftrightarrow \quad d_i = a_i - b_i \quad i = 1, \ldots, n$$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$a + b = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \qquad a - b = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

# multiplication by a scalar

Multiplication by a scalar involves multiplying each element in the vector by the scalar:

$$b = \sigma a \quad \Longleftrightarrow \quad b_i = \sigma a_i \quad i = 1, \ldots, n$$

$$a = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \qquad b = \frac{a}{2} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

# vector transpose

The *transpose* of a row vector is a column vector:

$$u = \begin{bmatrix} 1, 2, 3 \end{bmatrix} \qquad \text{then} \qquad u^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Likewise if $v$ is the column vector

$$v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad \text{then} \qquad v^T = \begin{bmatrix} 4, 5, 6 \end{bmatrix}$$

# linear combinations

Combine scalar multiplication with addition

$$\alpha \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \beta \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \\ \vdots \\ \alpha u_m + \beta v_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

$$r = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \qquad s = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$t = 2r + 3s = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 15 \end{bmatrix}$$

# linear combinations

Any one vector can be created from an infinite combination of other "suitable" vectors.

$$w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$w = 6 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

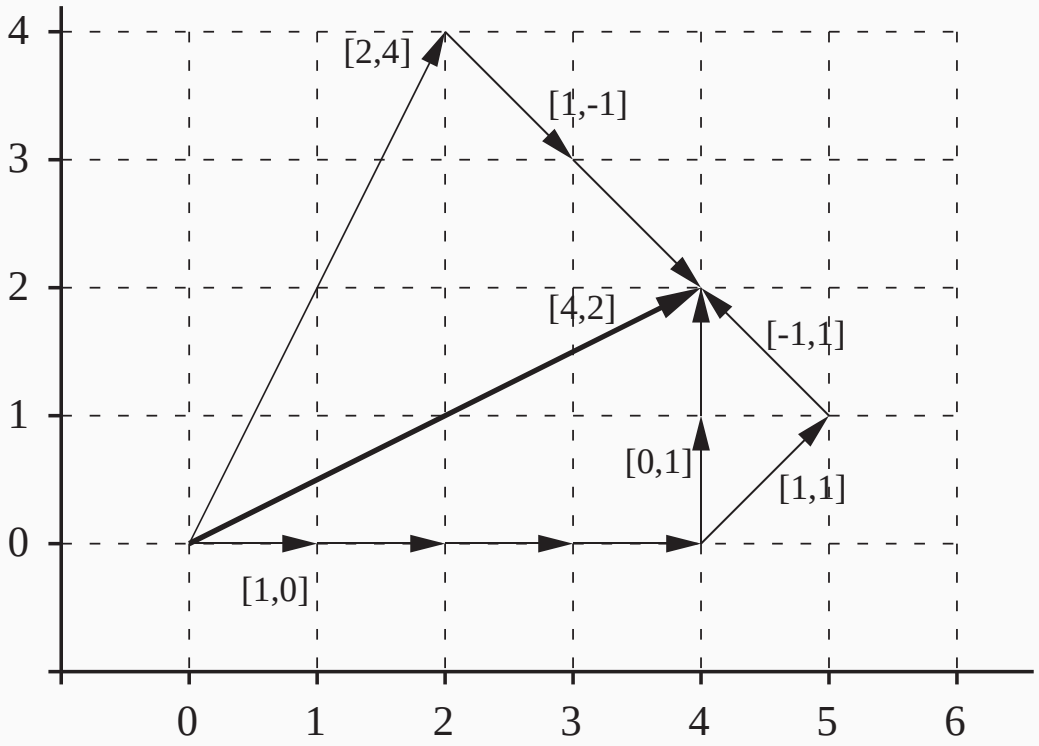
$$w = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$w = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

# linear combinations

**Graphical interpretation:**

- Vector tails can be moved to convenient locations
- Magnitude and direction of vectors is preserved



# vector inner product

In physics, analytical geometry, and engineering, the **dot product** has a geometric interpretation

$$\sigma = x \cdot y \quad \Longleftrightarrow \quad \sigma = \sum_{i=1}^{n} x_i y_i$$

$$x \cdot y = \|x\|_2 \, \|y\|_2 \cos \theta$$

# vector inner product

The inner product of $x$ and $y$ *requires* that $x$ be a row vector $y$ be a column vector

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$

# vector inner product

For two $n$-element *column* vectors, $u$ and $v$, the inner product is

$$\sigma = u^T v \quad \Longleftrightarrow \quad \sigma = \sum_{i=1}^{n} u_i v_i$$

The inner product is commutative so that
(for two column vectors)

$$u^T v = v^T u$$

# vector outer product

The inner product results in a scalar.
The *outer product* creates a rank-one matrix:

$$A = uv^T \quad \Longleftrightarrow \quad a_{i,j} = u_i v_j$$

## Example

Outer product of two 4-element column vectors

$$uv^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 & u_1 v_4 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 & u_2 v_4 \end{bmatrix}$$

# vector norms

Compare magnitude of scalars with the *absolute value*

$$|\alpha| > |\beta|$$

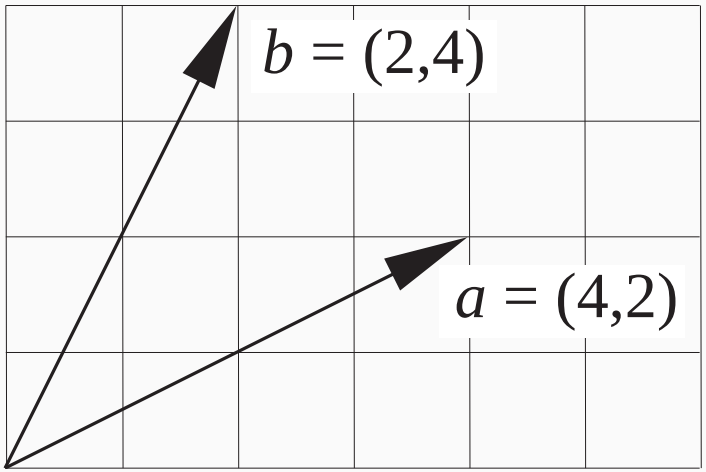
Compare magnitude of vectors with *norms*

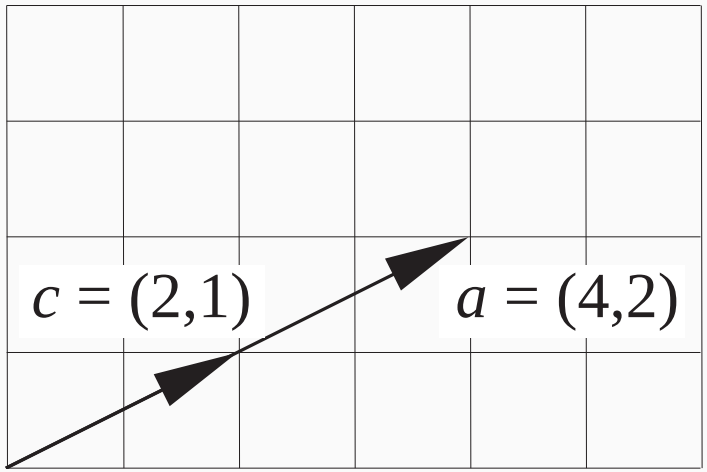
$$\|x\| > \|y\|$$

There are several ways to compute $\|x\|$. In other words the size of two vectors can be compared with different norms.

# vector norms

Consider two element vectors, which lie in a plane




Use geometric lengths to represent the magnitudes of the vectors

$$\ell_a = \sqrt{4^2 + 2^2} = \sqrt{20}, \qquad \ell_b = \sqrt{2^2 + 4^2} = \sqrt{20}, \qquad \ell_c = \sqrt{2^2 + 1^2} = \sqrt{}$$

We conclude that

$$\ell_a = \ell_b \quad \text{and} \quad \ell_a > \ell_c$$

or

$$\|a\| = \|b\| \quad \text{and} \quad \|a\| > \|c\|$$

# the $l_2$ norm

The notion of a geometric length for 2D or 3D vectors can be extended vectors with arbitrary numbers of elements.

The result is called the *Euclidian* or $L_2$ norm:

$$\|x\|_2 = \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

The $L_2$ norm can also be expressed in terms of the inner product

$$\|x\|_2 = \sqrt{x \cdot x} = \sqrt{x^T x}$$

# $p$-norms

For any positive integer $p$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{1/p}$$

The $L_1$ norm is sum of absolute values

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n| = \sum_{i=1}^{n} |x_i|$$

The $L_\infty$ norm or *max norm* is

$$\|x\|_\infty = \max\left(|x_1|, |x_2|, \ldots, |x_n|\right) = \max_i \left(|x_i|\right)$$

Although $p$ can be any positive number, $p = 1, 2, \infty$ are most commonly used.

# defining a $p$-norm

These must hold for any $x$ and $y$

1. $\|x\| > 0$ if $x \neq 0$
2. $\|\alpha x\| = |\alpha| \cdot \|x\|$ for an scalar $\alpha$
3. $\|x + y\| \leqslant \|x\| + \|y\|$ (this is called the triangle inequality)

# defining a $p$-norm for a matrix

If $A$ is a matrix, then we use the vector $p$-norm to define a similar matrix norm:

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

# application of norms

**Are two vectors (nearly) equal?**

Floating point comparison of two scalars with absolute value:

$$\frac{|\alpha - \beta|}{|\alpha|} < \delta$$

where $\delta$ is a small tolerance.

Comparison of two vectors with norms:

$$\frac{\|y - z\|}{\|z\|} < \delta$$

# application of norms

Notice that

$$\frac{\|y - z\|}{\|z\|} < \delta$$

is **not equivalent to**

$$\frac{\|y\| - \|z\|}{\|z\|} < \delta.$$

This comparison is important in convergence tests for sequences of vectors.

# application of norms

**Creating a Unit Vector**

Given $u = [u_1, u_2, \ldots, u_m]^T$, the unit vector in the direction of $u$ is

$$\hat{u} = \frac{u}{\|u\|_2}$$

Proof:

$$\|\hat{u}\|_2 = \left\| \frac{u}{\|u\|_2} \right\|_2 = \frac{1}{\|u\|_2} \|u\|_2 = 1$$

The following are *not* unit vectors

$$\frac{u}{\|u\|_1} \qquad \frac{u}{\|u\|_\infty}$$

# orthogonal vectors

From geometric interpretation of the inner product

$$u \cdot v = \|u\|_2 \, \|v\|_2 \cos\theta$$

$$\cos\theta = \frac{u \cdot v}{\|u\|_2 \, \|v\|_2} = \frac{u^T v}{\|u\|_2 \, \|v\|_2}$$

Two vectors are orthogonal when $\theta = \pi/2$ or $u \cdot v = 0$.

In other words

$$u^T v = 0$$

*if and only if* $u$ and $v$ are *orthogonal*.

# orthonormal vectors

**Orthonormal vectors** are **unit vectors** that are *orthogonal*.

A **unit** vector has an $L_2$ norm of one.

The unit vector in the direction of $u$ is

$$\hat{u} = \frac{u}{\|u\|_2}$$

Since

$$\|u\|_2 = \sqrt{u \cdot u}$$

it follows that $u \cdot u = 1$ if $u$ is a unit vector.

# notation

The matrix $A$ with $m$ rows and $n$ columns looks like:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

$a_{ij} =$ element in **row** $i$, and **column** $j$

# matrices consist of row and column vectors

As a collection of column vectors

$$A = \left[\begin{array}{c|c|c|c} a_{(1)} & a_{(2)} & \cdots & a_{(n)} \end{array}\right]$$

As a collection of row vectors

$$A = \left[\begin{array}{c} a'_{(1)} \\ \hline a'_{(2)} \\ \hline \vdots \\ \hline a'_{(m)} \end{array}\right]$$

A prime is used to designate a row vector on this and the following pages.

# preview of the row and column view

Matrix and
vector operations

$\updownarrow$

Row and column
operations

$\updownarrow$

Element-by-element
operations

# matrix operations

**Addition and subtraction**

$$C = A + B$$

or

$$c_{i,j} = a_{i,j} + b_{i,j} \quad i = 1, \ldots, m; \quad j = 1, \ldots, n$$

**Multiplication by a Scalar**

$$B = \sigma A$$

or

$$b_{i,j} = \sigma a_{i,j} \quad i = 1, \ldots, m; \quad j = 1, \ldots, n$$

## Note

Commas in subscripts are necessary when the subscripts are assigned numerical values. For example, $a_{2,3}$ is the row 2, column 3 element of matrix $A$, whereas $a_{23}$ is the $23\mathrm{rd}$ element of vector $a$. When variables appear in indices, such as $a_{ij}$ or $a_{i,j}$, the comma is

# matrix transpose

$$B = A^T$$

or

$$b_{i,j} = a_{j,i} \quad i = 1, \ldots, m; \quad j = 1, \ldots, n$$

# matrix–vector product

- The Column View
  - gives mathematical insight
- The Row View
  - easy to do by hand
- The Vector View
  - A square matrix rotates and stretches a vector

# column view of matrix–vector product

Consider a **linear combination of a set of column vectors** $\{a_{(1)}, a_{(2)}, \ldots, a_{(n)}\}$. Each $a_{(j)}$ has $m$ elements

Let $x_i$ be a set (a vector) of scalar multipliers

$$x_1 a_{(1)} + x_2 a_{(2)} + \ldots + x_n a_{(n)} = b$$

or

$$\sum_{j=1}^{n} a_{(j)} x_j = b$$

Expand the (hidden) row index

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

# column view of matrix–vector product

Form a matrix with the $a_{(j)}$ as columns

$$\left[\begin{array}{c|c|c|c} a_{(1)} & a_{(2)} & \cdots & a_{(n)} \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

Or, writing out the elements

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

# column view of matrix–vector product

Thus, the matrix-vector product is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Save space with matrix notation

$$Ax = b$$

# column view of matrix–vector product

**The matrix–vector product $b = Ax$ produces a vector $b$ from a linear combination of the columns in $A$.**

$$b = Ax \quad \Longleftrightarrow \quad b_i = \sum_{j=1}^{n} a_{ij} x_j$$

where $x$ and $b$ are column vectors

# column view of matrix–vector product

Listing 1: Matrix–Vector Multiplication by Columns

```
initialize:   b = zeros(m, 1)
for  j = 1, ..., n
   for  i = 1, ..., m
      b(i) = A(i, j)x(j) + b(i)
   end
end
```

# compatibility requirement

**Inner dimensions must agree**

$$\begin{array}{cccc} A & x & = & b \\ [m \times n] & [n \times 1] & = & [m \times 1] \end{array}$$

# row view of matrix–vector product

Consider the following matrix–vector product written out as a linear combination of matrix columns

$$\begin{bmatrix} 5 & 0 & 0 & -1 \\ -3 & 4 & -7 & 1 \\ 1 & 2 & 3 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -3 \\ -1 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ -7 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$$

This is the column view.

# row view of matrix–vector product

Now, group the multiplication and addition operations by row:

$$4\begin{bmatrix}5\\-3\\1\end{bmatrix}+2\begin{bmatrix}0\\4\\2\end{bmatrix}-3\begin{bmatrix}0\\-7\\3\end{bmatrix}-1\begin{bmatrix}-1\\1\\6\end{bmatrix}$$

$$=\begin{bmatrix}(5)(4) & + & (0)(2) & + & (0)(-3) & + & (-1)(-1)\\(-3)(4) & + & (4)(2) & + & (-7)(-3) & + & (1)(-1)\\(1)(4) & + & (2)(2) & + & (3)(-3) & + & (6)(-1)\end{bmatrix}=\begin{bmatrix}21\\16\\-7\end{bmatrix}$$

Final result is identical to that obtained with the column view.

# row view of matrix–vector product

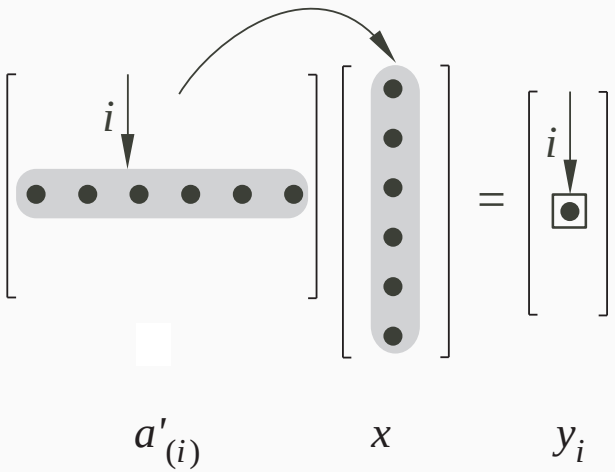
Product of a $3 \times 4$ matrix, $A$, with a $4 \times 1$ vector, $x$, looks like

$$\begin{bmatrix} a'_{(1)} \\ \hline a'_{(2)} \\ \hline a'_{(3)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a'_{(1)} \cdot x \\ a'_{(2)} \cdot x \\ a'_{(3)} \cdot x \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where $a'_{(1)}$, $a'_{(2)}$, and $a'_{(3)}$, are the *row vectors* constituting the $A$ matrix.

**The matrix–vector product $b = Ax$ produces elements in $b$ by forming inner products of the rows of $A$ with $x$.**

# row view of matrix–vector product



# vector view of matrix–vector product

If $A$ is square, the product $Ax$ has the effect of stretching and rotating $x$.

Pure stretching of the column vector

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Pure rotation of the column vector

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

# vector–matrix product

## Matrix–vector product

$$\underset{m\times n}{\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}}$$

## Vector–Matrix product

$$\underset{1\times m}{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}} \underset{m\times n}{\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}} = \underset{1\times n}{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}}$$

# vector–matrix product

**Compatibility Requirement: Inner dimensions must agree**

$$\begin{array}{cccc} u & A & = & v \\ [1 \times m] & [m \times n] & = & [1 \times n] \end{array}$$

# matrix–matrix product

Computations can be organized in **six different ways** We'll focus on just two

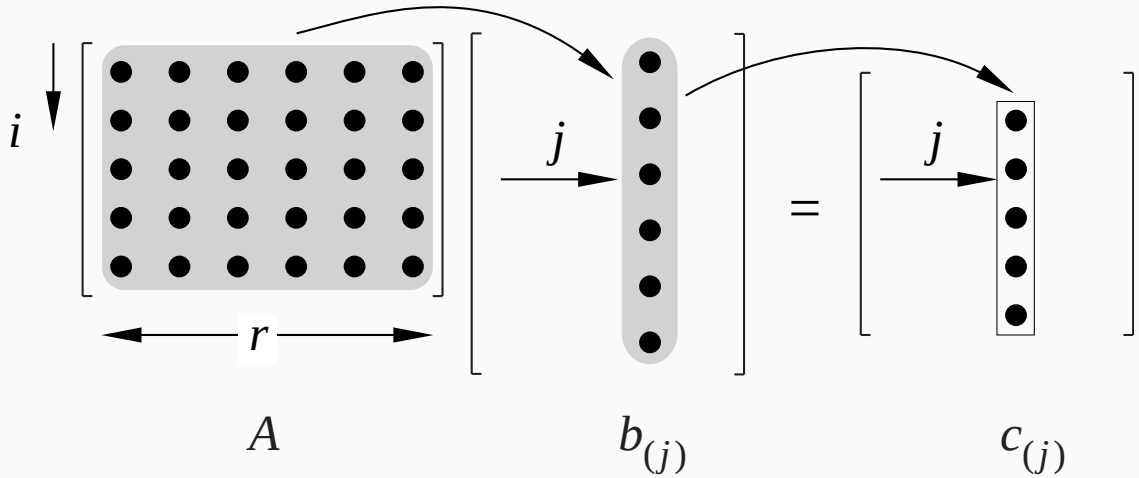
- Column View — extension of column view of matrix–vector product
- Row View — inner product algorithm, extension of column view of matrix–vector product

# column view of matrix–matrix product

The product $AB$ produces a matrix $C$. The columns of $C$ are linear combinations of the columns of $A$.

$$AB = C \qquad \Longleftrightarrow \qquad c_{(j)} = Ab_{(j)}$$

$c_{(j)}$ and $b_{(j)}$ are column vectors.



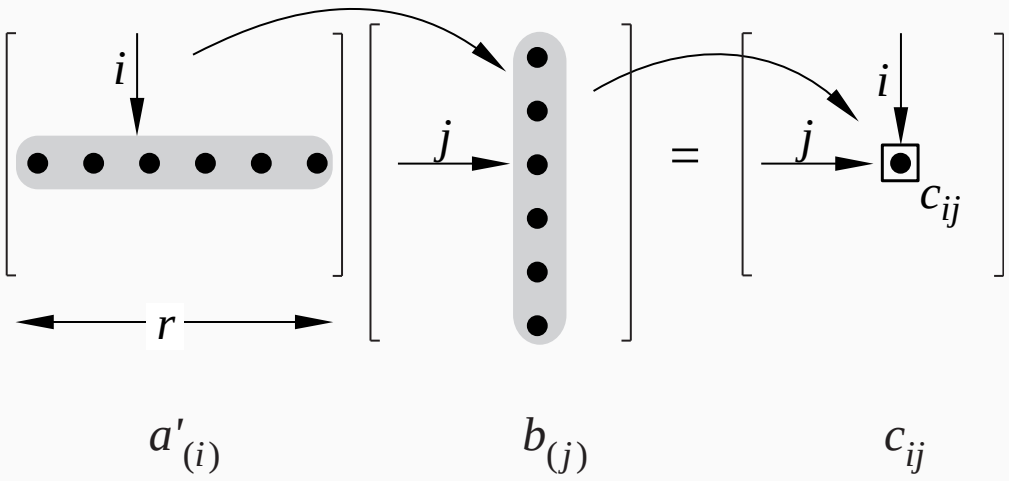

The column view of the matrix–matrix product $AB = C$ is helpful because it shows the relationship between the columns of $A$ and the columns of $C$.

# inner product (row) view of matrix–matrix product

The product $AB$ produces a matrix $C$. The $c_{ij}$ element is the *inner product* of row $i$ of $A$ and column $j$ of $B$.

$$AB = C \qquad \Longleftrightarrow \qquad c_{ij} = a'_{(i)} b_{(j)}$$

$a'_{(i)}$ is a row vector, $b_{(j)}$ is a column vector.




The inner product view of the matrix–matrix product is easier to use for hand calculations.

# matrix–matrix product summary

The **Matrix–vector product** looks like:

$$\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}$$

The **vector–Matrix product** looks like:

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet \end{bmatrix}$$

# matrix–matrix product summary

The **Matrix–Matrix product** looks like:

$$\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

# matrix–matrix product summary

**Compatibility Requirement**

$$\begin{array}{cccc} A & B & = & C \\ [m \times r] & [r \times n] & = & [m \times n] \end{array}$$

Inner dimensions must agree

Also, in general

$$AB \neq BA$$

# linear independence

Two vectors lying along the same line are not independent

$$u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad v = -2u = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$$

Any two independent vectors, for example,

$$v = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} \qquad \text{and} \qquad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

define a plane. Any other vector in this plane of $v$ and $w$ can be represented by

$$x = \alpha v + \beta w$$

$x$ is **linearly dependent** on $v$ and $w$ because it can be formed by a linear combination of $v$ and $w$.

# linear independence

A set of vectors is linearly independent if it is impossible to use a linear combination of vectors in the set to create another vector in the set.

Linear independence is easy to see for vectors that are orthogonal, for example,

$$\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent.

# linear independence

Consider two linearly independent vectors, $u$ and $v$.

If a third vector, $w$, *cannot* be expressed as a linear combination of $u$ and $v$, then the set $\{u, v, w\}$ is linearly independent.

In other words, if $\{u, v, w\}$ is linearly independent then

$$\alpha u + \beta v = \delta w$$

can be true *only if* $\alpha = \beta = \delta = 0$.

More generally, if the only solution to

$$\alpha_1 v_{(1)} + \alpha_2 v_{(2)} + \cdots + \alpha_n v_{(n)} = 0 \tag{1}$$

is $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, then the set $\{v_{(1)}, v_{(2)}, \ldots, v_{(n)}\}$ is **linearly independent**. Conversely, if equation (1) is satisfied by at least one nonzero $\alpha_i$, then the set of vectors is **linearly dependent**.

# linear independence

Let the set of vectors $\{v_{(1)}, v_{(2)}, \ldots, v_{(n)}\}$ be organized as the columns of a matrix. Then the condition of linear independence is

$$\left[\begin{array}{c|c|c|c} & & & \\ v_{(1)} & v_{(2)} & \cdots & v_{(n)} \\ & & & \end{array}\right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2}$$

**The columns of the $m \times n$ matrix, $A$, are linearly independent if and only if $x = (0, 0, \ldots, 0)^T$ is the only $n$ element column vector that satisfies $Ax = 0$.**

# spaces and subspaces

Group vectors according to number of elements they have. Vectors from these different groups cannot be mixed.

$\mathbf{R}^1$ = Space of all vectors with one element.

These vectors define the points along a line.

$\mathbf{R}^2$ = Space of all vectors with two elements.

These vectors define the points in a plane.

$\mathbf{R}^n$ = Space of all vectors with $n$ elements.

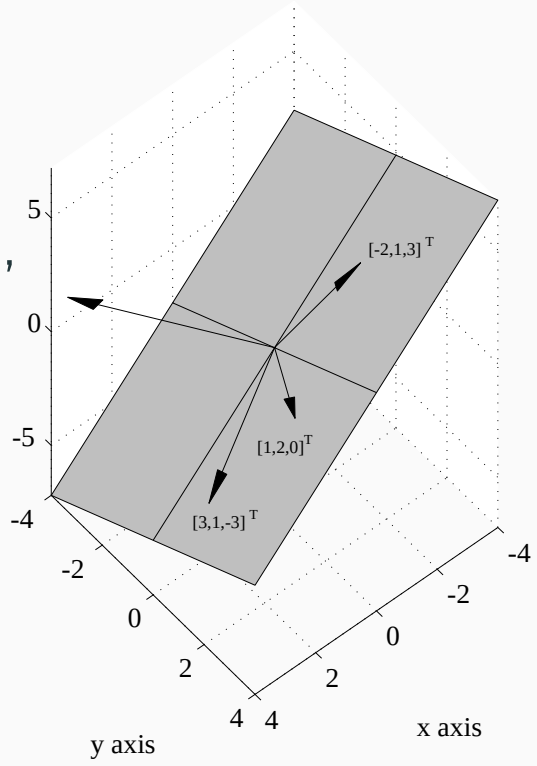
These vectors define the points in an $n$-dimensional space (hyperplane).

# subspaces

The three vectors

$$u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \quad w = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix},$$

lie in the same plane. The vectors have three elements each, so they belong to $\mathbf{R}^3$, but they **span** a **subspace** of $\mathbf{R}^3$.



# basis and dimension of a subspace

- A **basis** for a subspace is a set of **linearly independent** vectors that **span** the subspace.
- Since a basis set must be linearly independent, it also must have the smallest number of vectors necessary to span the space. (Each vector makes a unique contribution to spanning some other direction in the space.)
- The number of vectors in a basis set is equal to the **dimension** of the **subspace** that these vectors span.
- Mutually orthogonal vectors (an orthogonal set) form convenient basis sets, but basis sets need not be orthogonal.

# subspaces associated with matrices

The matrix–vector product

$$y = Ax$$

creates $y$ from a linear combination of the columns of $A$

The column vectors of $A$ form a basis for the **column space** or **range** of $A$.

# matrix rank

- The **rank** of a matrix, $A$, is the number of linearly independent columns in $A$.
- $\mathrm{rank}(A)$ is the dimension of the column space of $A$.
- Numerical computation of $\mathrm{rank}(A)$ is tricky due to roundoff.

Consider

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad w = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Do these vectors span $\mathbf{R}^3$?

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Consider

$$u = \begin{bmatrix} 1 \\ 0 \\ 0.00001 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad w = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Do these vectors span $\mathbf{R}^3$?

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Consider

$$u = \begin{bmatrix} 1 \\ 0 \\ \varepsilon_m \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad w = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Do these vectors span $\mathbf{R}^3$?