#7

Linear Algebra Meets Computation

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focus of the day

- why linear algebra and computation?
- how do we represent data in a linear algebra form?
- build a connection between data and vectors
- investigate operations on vectors (i.e. data)
- look ahead toward more sophisticated operators on data
why linear algebra?

• what connection does linear algebra have with numerics?
• what math operations can we perform on a computer? (think FPU).
• linear algebra. specifically:
  • vectors $\rightarrow$ data
  • matrices $\rightarrow$ operators on data
vector addition and subtraction

Addition and subtraction are element-by-element operations

\[ c = a + b \iff c_i = a_i + b_i \quad i = 1, \ldots, n \]
\[ d = a - b \iff d_i = a_i - b_i \quad i = 1, \ldots, n \]

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\quad \begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix}
\]

\[
a + b = \begin{bmatrix}
4 \\
4 \\
4
\end{bmatrix}
\quad a - b = \begin{bmatrix}
-2 \\
0 \\
2
\end{bmatrix}
\]
Multiplication by a scalar involves multiplying each element in the vector by the scalar:

\[ b = \sigma a \quad \iff \quad b_i = \sigma a_i \quad i = 1, \ldots, n \]

\[
\begin{align*}
a &= \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \\
b &= \frac{a}{2} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}
\end{align*}
\]
linear combinations

Combine scalar multiplication with addition

$$\alpha \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \beta \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \\ \vdots \\ \alpha u_m + \beta v_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

$$r = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \quad s = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$t = 2r + 3s = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 15 \end{bmatrix}$$
linear combinations

Any one vector can be created from an infinite combination of other “suitable” vectors.

\[ w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ w = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

\[ w = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

\[ w = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
why do these operations make sense?
linear independence and a basis

• A set of vectors \( \{u_1, u_2, \ldots, u_m\} \) are said to be **linearly independent** if

\[
\sum_{i=1}^{m} \alpha_i u_i = 0 \text{ only when } \alpha_i = 0 \quad \forall i
\]

Otherwise the set is **linearly dependent**.

• A **basis** is a set of linearly independent vectors, such that any other vector is a linear combination of the basis vectors.
Graphical interpretation:

- Vector tails can be moved to convenient locations.
- Magnitude and direction of vectors is preserved.
linear, affine, and convex combinations

linear:

\[ \sum_{i=1}^{n} \alpha_i u_i \quad \alpha_i \in \mathbb{R} \quad u_i \in \mathbb{R}^m \]

affine:

same as linear with the added constraint:

\[ \sum_{i=1}^{n} \alpha_i = 1 \]

convex:

same as affine with the added constraint:

\[ \alpha_i > 0 \quad \forall i \]
The *transpose* of a row vector is a column vector:

\[ u = [1, 2, 3] \quad \text{then} \quad u^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \]

Likewise if \( v \) is the column vector

\[ v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{then} \quad v^T = [4, 5, 6] \]
In physics, analytical geometry, and engineering, the **dot product** has a geometric interpretation.

\[ \sigma = x \cdot y \iff \sigma = \sum_{i=1}^{n} x_i y_i \]

\[ x \cdot y = \|x\|_2 \|y\|_2 \cos \theta \]
The inner product of $x$ and $y$ \textit{requires} that $x$ be a row vector $y$ be a column vector

$$
\begin{bmatrix}
x_1 & x_2 & x_3 & x_4
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix}
= x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4
$$
For two \( n \)-element *column* vectors, \( u \) and \( v \), the inner product is

\[
\sigma = u^T v \quad \iff \quad \sigma = \sum_{i=1}^{n} u_i v_i
\]

The inner product is commutative so that

(for two column vectors)

\[
u^T v = v^T u
\]
The inner product results in a scalar.

The *outer product* creates a rank-one matrix:

\[ A = uv^T \iff a_{i,j} = u_i v_j \]
Vectors (i.e. data) is one-half of our Linear Algebra. The other focuses on *Operators* acting on the vectors. What can these operators do?

- Scaling
- Permutations
- Rotation
- Used in Linear System Solves

Do they (*Operators*) have another name?

- Matrices
The operator $A$ with $m$ rows and $n$ columns looks like:

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & \cdots & & a_{mn}
\end{bmatrix}$$

$a_{ij} = \text{element in row } i, \text{ and column } j$

Recall the significance of the entry position for a vector (e.g. if $a = [1, 2, 4, 9, \ldots]$, what is the meaning of 1 in the first entry). What is the significance of the entry positions in $A$ then?
matrices consist of row and column vectors

As a collection of column vectors

\[ A = \begin{bmatrix}
    a_1 & a_2 & \cdots & a_n
\end{bmatrix} \]

As a collection of row vectors

\[ A = \begin{bmatrix}
    a'_1 \\ a'_2 \\ \vdots \\ a'_m
\end{bmatrix} \]
some remarks

- data can be represented by vectors in the linear algebra sense
- we have seen how to perform vector operations
- we will see *operators* can be applied to the data to yield more interesting and useful results
- linear algebra forms the base of our numerical methods