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Linear Algebra Meets Computation (cont.)

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goals for today

- Last time we learned about representing data; now how can we manipulate it?
- Thinking about a matrix as an operator on data
- Using an operator to rotate, scale, etc. geometric data
- Applying these operators to actual problems

recall matrix-vector multiplication

Consider computing $Ax = y$.

- Recall summation notation

$$y_i = \sum_j A_{ij} x_j$$

- Not very intuitive
- Two ways to think about mat-vecs:
  - Linear combination of column vectors
  - Dot product of x with rows of A

recall matrix-matrix multiplication

Consider computing $AB = C$.

- Recall summation notation

$$C_{ij} = \sum_k A_{ik} B_{kj}$$

- Actually just applying mat-vec to lots of column vectors

matrices as operators

- Matrices operate on data
- For $y = Ax$, x is transformed into y
- Data can be list of values (stored in a vector) or geometric vectors

matrices operating on data

- Recall representing data as vectors, e.g. sound
- How could we represent an averaging operation?
- Recall representing data as matrices (2d vectors), e.g. image
- Image blurring demo

matrices operating on vectors

What can matrices do?

- Scale
- Rotate
- Can they translate? – Think on this

scaling operator

- Stretching an image or a vector
- Can stretch by different amounts in different dimensions
- Example:

$$\begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & c_m \end{bmatrix}$$

- Scaling by negative number reflects over axis

rotating operator

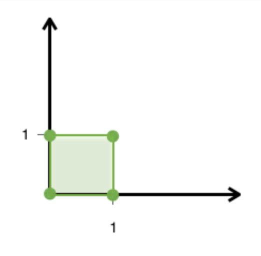
- Think about rotating $[1, 0]^T$ and $[0, 1]^T$ clockwise by θ
- What do these vectors turn into?
- Example:

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

- What changes to rotate counter-clockwise?
- Matrices for geometry transformation demo

can matrices translate?

- Consider a 2×2 matrix A
- Suppose we have 4 points representing the corners of the unit square:



- Is it possible to mulitply each point by A (i.e. Ax for each point x) to move (translate) this square?

special matrices

- Identity matrix: operator that does nothing

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

- Permutation matrix
  - Permutation of the identity matrix
  - Permutes (swaps) rows
  - Example:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ a \\ b \end{bmatrix}$$

matrices are linear operators

What do we mean by linear?

$$A(x + y) = Ax + Ay$$

$$A(\alpha x) = \alpha Ax$$

matrices as sets of vectors

We have been thinking about matrices as operating on vectors. It can also be useful to think of matrix as a set of column vectors.

But first, let's revisit some topics from last time:

- Recall linear independence/dependence: A set of vectors $v_1, v_2, \ldots, v_m$ are linearly independent if
$$\sum_{i=1}^{m} \alpha_i v_i = 0 \iff \alpha_i = 0 \forall i$$
Otherwise set is linearly dependent.
- Recall basis: A set of linearly independent vectors such that any other vectors in the same space can be represented as a linear combination of the basis vectors.
- Span: Set of vectors that can be written as a linear combination of basis vectors

matrices as sets of vectors (cont).

So, what does this have to do with matrices?

Consider a matrix A, composed of column vectors, $a_1, a_2, \ldots, a_m$

- Rank(A): the number of linearly independent columns
  - Column rank = row rank
- Nullspace(A): the set of all vectors that A annihilates – any x such that $Ax = 0$
  - Zero vector always in the nullspace of a matrix
  - What does it mean for nullspace to be nontrivial?
  - Connection to rank?
- Range(A): the span of the columns of A – all vectors y such that $Ax = y$ for some x (think about mat-vec as linear combination of columns)

linear systems

- Matvec is computing $y = Ax$.
- What if we know y, but not x?
- In terms of matrices as sets of vectors, think of this as finding the coefficients to the vectors so linear combination produces right hand side.
- In terms of matrices as linear operators, think of this as finding data that when operated on by equations gives right hand side.

linear systems (cont)

Example: Consider linear system:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 5 \\ 4x_1 + x_2 + x_3 &= 9 \\ x_1 - x_2 + 3x_3 &= 8 \end{aligned}$$

This can be represented as

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 8 \end{bmatrix}$$

The solution, x to $Ax = y$ is

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

What does this represent in both views from last slide?