Overview

1. Properties of mat, norm

   - Condition number of a matrix

   - Solve

\[ \| x \|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \]

\[ \| A \|_2 = \max_{\| x \|_2 = 1} \frac{\| A x \|_2}{\| x \|_2} \]

Recup mat, norm
Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. \( \|A\| > 0 \iff A \neq 0 \).
2. \( \|\gamma A\| = |\gamma| \|A\| \) for all scalars \( \gamma \).
3. Obeys triangle inequality \( \|A + B\| \leq \|A\| + \|B\| \)

But also some more properties that stem from our definition:

\[
\|Ax\| \leq \|A\| \|x\|
\]

\[
\|AB\| \leq \|A\| \|B\|
\]

\[
\sup_{\|x\|=1} \frac{\|Ax\|}{\|x\|} \leq \|A\|
\]

prop. of a vector norm
Example: Orthogonal Matrices

- What is the 2-norm of an orthogonal matrix?

\[ \| A \|_2 = \max \| A x \|_2 = \max \sqrt{\sum (A x)_i^2} = 1 \]

\[ \| A x \|_2 = \sqrt{\sum (A x)^2} = \sqrt{\| A^T A x \|_2} = \sqrt{\| x \|_2} \]

\[ A^T A = \text{diagonal} \]

\[ A^T A = I \]

\[ A \text{ orthogonal} \]

\[ A^T = A \]

\[ A B = I \]

\[ B A = I \]

\[ A^T = A^{-1} \]
Conditioning

- Now, let’s study condition number of solving a linear system

\[ Ax = b. \]
Upper bound on the condition number

... with more work; actually "sharp"

\[ \text{best upper bound you can get} \]

\[ \text{cond}_2 (A) = \| A \|_2 \cdot \| A^{-1} \|_2 \]

\[ \text{cond}_\infty (A) = \| A \|_\infty \cdot \| A^{-1} \|_\infty \]

Condition number of a matrix is always relative to a given matrix norm.

If \( A^{-1} \) doesn't exist, then

\[ A \neq b \quad \text{by convention, } \text{cond}(A) = \infty \]
Demo: Condition number visualized
Demo: Conditioning of $2 \times 2$ Matrices
Matrices with Great Conditioning (Part 1)

- Give an example of a matrix that is very well-conditioned. (I.e. has a condition-number that’s good for computation.)
  What is the best possible condition number of a matrix?
What is the 2-norm condition number of an orthogonal matrix $A$?
In-class activity: Matrix Conditioning