Overview

- num. diff / finite differences
- num. int = quadrature
- eigen values
Demo: Taking derivatives with Vandermonde matrices

\[ p(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n \]

\[ p'(x) = \alpha_1 (0, x, 1) + \alpha_2 (0, x^2, 2) + \ldots + \alpha_n (0, x^n, n) \]

\[ p''(x) = \alpha_1 \cdot p'(x) + \alpha_2 \cdot p''(x) + \ldots + \alpha_n \cdot p''(x) \]

\[ q(x) = \beta_0 \cdot \psi_0(x) + \beta_1 \cdot \psi_1(x) + \ldots + \beta_n \cdot \psi_n(x) \]

\[
\begin{bmatrix}
\frac{\partial^2 q}{\partial x^2} \\
\end{bmatrix} = V' \cdot V^{-1} \cdot \partial^2 \cdot V^{-1} = V^{-1} \cdot \partial^2 \cdot V^{-1} = (V' \cdot V^{-1}) \cdot (V' \cdot V^{-1}) \cdot \partial^2 \cdot V^{-1} \]

\[
\begin{bmatrix}
\frac{\partial^2 q}{\partial x^2} \\
\end{bmatrix} = V^{-1} \cdot \partial^2 \cdot V^{-1} = V^{-1} \cdot \partial^2 \cdot V^{-1} \]

\[ \hat{D}_2 = \begin{bmatrix}
\partial^2 \\
\end{bmatrix} = V^{-1} \cdot \partial^2 \cdot V^{-1} \]
Finite Difference Formulas

- It is possible to use the process above to find ‘canned’ formulas for taking derivatives. Suppose we use three points equispaced points \((x - h, x, x + h)\) for interpolation (i.e. a degree-2 polynomial).
  - What is the resulting differentiation matrix?
  - What does it tell us?

\[
\begin{aligned}
D &= V^{-1} V^T \\
D \begin{pmatrix}
  f(x-h) \\
  f(x) \\
  f(x+h)
\end{pmatrix} &\approx \begin{pmatrix}
  \frac{f(x-h) - f(x)}{-2h} \\
  \frac{f(x) - f(x+h)}{2h} \\
  \frac{f(x+h) - f(x-h)}{2h}
\end{pmatrix}
\end{aligned}
\]
\[ \sum_{i=1}^{\infty} \left( f'(x_i) \times \frac{f(x_i + h_i) - f(x_i, \ldots, x_n)}{h_i} \right) \]
Computing Integrals with Interpolation

- Can we use a similar process to compute (approximate) integrals of a function $f$?

\[
\int_a^b f(x) \, dx \approx \int_a^b \sum_{i=0}^n c_i \varphi_i(x) \, dx = \sum_{i=0}^n c_i \int_a^b \varphi_i(x) \, dx
\]

To compute $\int_a^b f(x) \, dx$, let $c_i = \int_a^b \varphi_i(x) \, dx$ for $n$ points.
Example: Building a Quadrature Rule

Demo: Computing the Weights in Simpson’s Rule

Suppose we know

\[ f(x_0) = 2 \quad f(x_1) = 0 \quad f(x_2) = 3 \]

\[ x_0 = 1 \quad x_1 = \frac{1}{2} \quad x_2 = 1 \]

How can we find an approximate integral?

\[ \int_0^1 x \, dx = \frac{1}{2} \]

\[ \int_0^1 x^2 \, dx = \left[ \frac{1}{3} x^3 \right]_0^1 = \frac{1}{3} \]

\[ \int_0^1 f(x) \, dx \approx \frac{1}{6} \cdot f(0) + \frac{4}{6} \cdot f\left(\frac{1}{2}\right) + \frac{1}{6} \cdot f(1) \]
Facts about Quadrature

- What does Simpson’s rule look like on $[0, 1/2]$?
- What does Simpson’s rule look like on $[5, 6]$?
- How accurate is Simpson’s rule?
  
  **Demo:** Accuracy of Simpson’s rule

10 Repeating Linear Operations: Eigenvalues and Steady States
Eigenvalue Problems: Setup/Math Recap

$A$ is an $n \times n$ matrix.

- $x \neq 0$ is called an eigenvector of $A$ if there exists a $\lambda$ so that
  \[ Ax = \lambda x. \]
- In that case, $\lambda$ is called an eigenvalue.