10 Repeating Linear Operations: Eigenvalues and Steady States
Eigenvalue Problems: Setup/Math Recap

$A$ is an $n \times n$ matrix.

- $\mathbf{x} \neq \mathbf{0}$ is called an eigenvector of $A$ if there exists a $\lambda$ so that $A\mathbf{x} = \lambda \mathbf{x}$.

- In that case, $\lambda$ is called an eigenvalue.
Finding Eigenvalues

- How do you find eigenvalues?

\[ A\mathbf{x} = \lambda \mathbf{x} \]

\[ (A - \lambda I)\mathbf{x} = \mathbf{0} \]

\[ (A - \lambda I) \text{ is a polynomial of degree } n \]

\[ \det(A - \lambda I) = 0 \]

\[ \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0 = 0 \]

Abel showed a polynomial like this with degree \( \geq 5 \) has no general formula for its roots.

\[ \Rightarrow \text{ no algorithm w/ finite number of steps} \]
Transforming Eigenvalue Problems

Suppose we know that \( A \mathbf{x} = \lambda \mathbf{x} \). What are the eigenvalues of these changed matrices?

- **Shift.** \( A \rightarrow A - \sigma I \)
  \[
  (A - \sigma I) \mathbf{x} = A \mathbf{x} - \sigma \mathbf{x} = \lambda \mathbf{x} - \sigma \mathbf{x} = (\lambda - \sigma) \mathbf{x}
  \]

- **Inversion.** \( A \rightarrow A^{-1} \)
  \[
  A^{-1} \mathbf{x} = \frac{1}{\lambda} \mathbf{x}^2
  \]

- **Power.** \( A \rightarrow A^k \)
  \[
  A^3 \mathbf{x} = A A A \mathbf{x} = A A \lambda \mathbf{x} = \lambda^3 \mathbf{x}
  \]

- **Inverse.** \( A \rightarrow A^{-1} \)
  \[
  A^{-3} \mathbf{x} = \frac{1}{\lambda} A^{-1} \mathbf{x}
  \]

**Polynomial**

\[
(A^3 + 5A^2 - 7A + 4) \mathbf{x}
\]

\[
= \lambda^3 + 5\lambda^2 - 7\lambda + 4 \mathbf{x}
\]

\[
= (\lambda^3 + 5\lambda^2 - 7\lambda + 4) \mathbf{x}
\]

\[
A_{1,1} \mathbf{x}_1 = \lambda_1 \mathbf{x}_1,
\]

\[
A_{n,n} \mathbf{x}_n = \lambda_n \mathbf{x}_n
\]

-1, 2, 5, -7
Changing Eigenvectors

- Suppose $Ax = \lambda x$.
  Can we change the eigenvectors? (but leave the eigenvalues the same)

$$\rightarrow \quad T^{-1}AT$$

$T^{-1}AT \tilde{y} = \lambda \tilde{y}$

$AT \tilde{y} = \lambda' T \tilde{y}$  infer: Use $\tilde{y} = T^{-1} \tilde{x}$

$\in A(T(T^{-1} \tilde{x})) = \lambda' T (T^{-1} \tilde{x})$

$\in A \tilde{x} = \lambda \tilde{x}$

The eigenvalues of $T^{-1}AT$ are the same as those of $A$, and for each eigenvector $\tilde{x}$ of $A$, $\tilde{y} = T^{-1} \tilde{x}$ is an ev of $T^{-1}AT$.
Diagonalizability

- When is a matrix called **diagonalizable**?

SIMPLIFIED:

If you can find a matrix $T$ so that $T^T A T = 0$, then $A$ and $B$ are called similar.

Suppose I had all the eigenvectors $x_i$:

$$A x_i = \lambda_i x_i$$

Then:

$$X = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$$

$$AX = X \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} X^{-1}$$

$$\Rightarrow X^{-1} A X = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \in \text{"diagonalize"}$$
Are all Matrices Diagonalizable?

- Give characteristic polynomial, eigenvalues, eigenvectors of

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \]

\[ \det (A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 \left( \begin{array}{c} \lambda = 1 \\ \lambda_2 = 1 \end{array} \right) \]

\[ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \implies u + v = u \sim v = 0 \]

\[ v = v \]

All eigenvectors are of the form \( \begin{pmatrix} u \\ 0 \end{pmatrix} \)

My eigenvector matrix \( X = \begin{pmatrix} u_1 & u_2 \\ 0 & 0 \end{pmatrix} \) is not invertible.
Power Iteration

- What are the eigenvalues of $A^{1000}$?

Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $x_1, \ldots, x_n$.
Further assume $\|x_i\| = 1$. 

Rayleigh quotient

$\frac{x_1^T A x_1}{x_1^T x_1} = \frac{x_1^T \lambda_1 x_1}{x_1^T x_1} = \lambda_1$
Power Iteration: Issues?

- What could go wrong with Power Iteration?

- $x_1 = 0$
- complex valued eigenvalues/eigenvectors
- $|\lambda_1| = |\lambda_2|$
What about Eigenvalues?

- Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

\[ \Rightarrow \text{Rayleigh quotient (see above)} \]

\[ \frac{x^T \mathbf{A} x}{x^T x} \]
Convergence of Power Iteration

- What can you say about the convergence of the power method?

  Say $v_1^{(k)}$ is the $k$th estimate of the eigenvector $x_1$, and $e_k = \| x_1 - v_1^{(k)} \|$. 