Overview

Low-rank approximation

Iteration

Equation solving
SVD as Sum of Outer Products

○ What’s another way of writing the SVD?

\[ A = U \Sigma V^\top = \begin{pmatrix} \sigma_1 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{pmatrix} \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \]

\[ \text{rank}(UV^\top) = 1 \]
\[ \sigma_i \hat{u}_i v_i^T + \sigma_j \hat{u}_j v_j^T + \ldots + \sigma_n \hat{u}_n v_n^T \]
Low-Rank Approximation (I)

- What is the *rank* of $\sigma_1 u_1 v_1^T$?

- What is the *rank* of $\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$?

**Demo:** Image Compression
Low-Rank Approximation

- What can we say about the low-rank approximation
  \[ A_k = \sigma_1 u_1 v_1^T + \cdots + \sigma_k u_k v_k^T \]
  to
  \[ A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T \]?

For simplicity, assume \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0 \).

Then among all rank-\( k \) matrices \( B \), \( A_k \) is the one that satisfies

\[ \min_{B} \| A \cdot B \|_F = \| A - A_k \|_F \]

Also:
\[ \| A - A_k \|_2 = \| \sigma_{k+1} u_{k+1} v_{k+1}^T + \cdots + \sigma_n u_n v_n^T \|_2 = \sigma_{k+1} \]
\[ \| A - A_h \|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_k^2} \]
Part 3: Approximation—When the Exact Answer is Out of Reach
14 Iteration and Convergence
What is linear convergence? quadratic convergence?

For power: \[ \|e_{k+1}\| = \left(\frac{\lambda_2}{\lambda_1}\right) \|e_k\| \]

\[ \|e_{k+1000}\| = \left(\frac{\lambda_2}{\lambda_1}\right)^{1000} \|e_k\| \subseteq \text{linear convergence} \]

- gains a fixed possibly fractional number of digits every time

\[ \|e_{k+1}\| \leq C \cdot \|e_k\| \subseteq \text{linear convergence} \]

\[ \|e_{k+1}\| < C \cdot \|e_k\|^2 \subseteq \text{quadratic convergence} \]

Example for quadratic:

\[ \|e_1\| = 0.1 \sim \frac{1}{10} \sim 10^{-1} \]

\[ \|e_2\| = 0.01 \sim \frac{1}{100} \sim 10^{-2} \]

\[ \|e_3\| = 10^{-4} \sim \frac{1}{10000} \sim 10^{-4} \]

\[ \|e_4\| (10^{-1})^2 = 10^{-2} \sim 10^{-2} \]
An iterative method converges with rate \( r \) if

\[
\lim_{k \to \infty} \frac{\|x_{k+1}\|}{\|x_k\|^r} = c < \infty
\]

converges with rate \( 1 : \text{linear} \)

\( 2 : \text{quadratic} \)
About Convergence Rates

Demo: Rates of Convergence

- Characterize linear, quadratic convergence in terms of the ‘number of accurate digits’.
15 Solving One Equation
Solving Nonlinear Equations

- What is the goal here?

Given: equation

\[ f(x) = 0 \]

Find \( x \) so that the equation is true.

\[ ax + b = 0 \]
\[ ax^2 + bx + c = 0 \]

\[ f(x) = y \]
\[ \hat{f}(x) = f(x) - y \]

solve

\[ \hat{f}(x) = 0 \]
Bisection Method

Demo: Bisection Method

- What's the rate of convergence? What's the constant?
Newton’s Method

- Derive Newton’s method.