2 Making Models with Polynomials

$[1, 2, 3]$

$\hookrightarrow$ vector space $\quad a, b$

$a + b \qquad \text{number} \cdot a$

assoc: $\quad (a+b)+c = a+(b+c),$

$(\alpha+\beta)\cdot c = \alpha\cdot c + \beta\cdot c$

18 multiple choice Q → 1pt

3 coding Q → 2pt

24pts / 22pts.

coding, CA, poly, Taylor, bigO

↳ Python, numpy docs available

Why polynomials?

$$a_3x^3 + a_2x^2 + a_1x + a_0$$

- How do we write the general case?
- *Why* polynomials and not something else?

Add + Multiply

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Reconstructing a Function From Derivatives

- If we know $f(x_0), f'(x_0), f''(x_0)$, can we reconstruct the function as a polynomial?

$$f(x) = {\color{red}???} + {\color{red}???}x + {\color{red}???}\,x^2 + \cdots$$

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i$$

$$= f(0) + \frac{f'(0)}{1} \cdot x$$

Demo: Polynomial Approximation with Polynomials (Part I)

Shifting the Expansion Center

- Can you do this at points other than the origin?

$$f(x) = \sum_{i=0} \frac{f^{(i)}(0)}{i!} x^i$$

$$0 \mapsto x_0$$

$$x \mapsto x - x_0$$

$$f(x_0 + x - x_0) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i$$

$$x - x_0 \mapsto h: \quad f(x_0 + h) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i$$

Errors in Taylor Approximation (I)

- Can't sum infinitely many terms. Have to truncate. How big of an error does this cause?

  Demo: Polynomial Approximation with Polynomials (Part II)

A Taylor expansion up to degree n

has error $O(h^{n+1})$.

$$\left| f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i \right| = O(h^{n+1})$$

Making Predictions with Taylor Truncation Error

- Suppose you expand $\sqrt{x-10}$ in a Taylor polynomial of degree 3 about the center $x_0 = 12$. For $h_1 = 0.5$, you find that the Taylor truncation error is about 10^{-4}.

  What is the Taylor truncation error for $h_2 = 0.25$?

$$10^{-4} = \text{Error}(h_1) = O(h_1^4) \approx C \cdot h^4$$

$$\text{Error}(h_2) = O(h_2^4) \approx C \cdot h^4$$

$$\text{Error}(h_2) = C \cdot h_2^4 = \underline{C \cdot h_1^4} \cdot \left(\frac{h_2}{h_1}\right)^4$$

$$= \text{Error}(h_1) \cdot \left(\frac{h_2}{h_1}\right)^4$$

Demo: Polynomial Approximation with Polynomials (Part III)