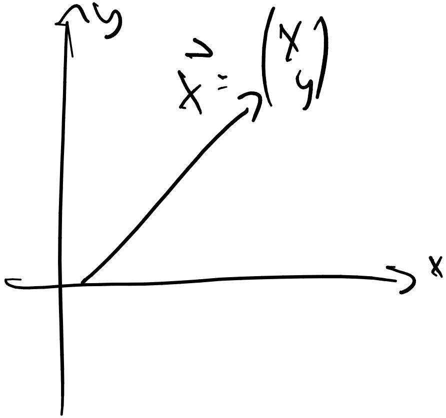
7 Norms and Errors

$$\|\vec{x}\|$$



$$\|\vec{x}\|_2 = \sqrt{x^2 + y^2}$$

$$\|\vec{x}\|_p = \sqrt[p]{|x|^p + |y|^p}$$

$$\|\vec{x}\|_\infty = \max(|x|, |y|)$$

Norms

- What's a norm?

- Define norm.

Examples of Norms

- What are some examples of norms?

Demo: Vector norms

$$\longrightarrow \|\alpha \vec{x}\| = |\alpha| \, \|x\|$$

Norms and Errors

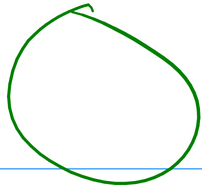
- If we're computing a vector result, the error is a vector. That's not a very useful answer to 'how big is the error'. What can we do?

Computed result = True result + error

$$\vec{x} = \vec{x}_0 + \Delta\vec{x}$$

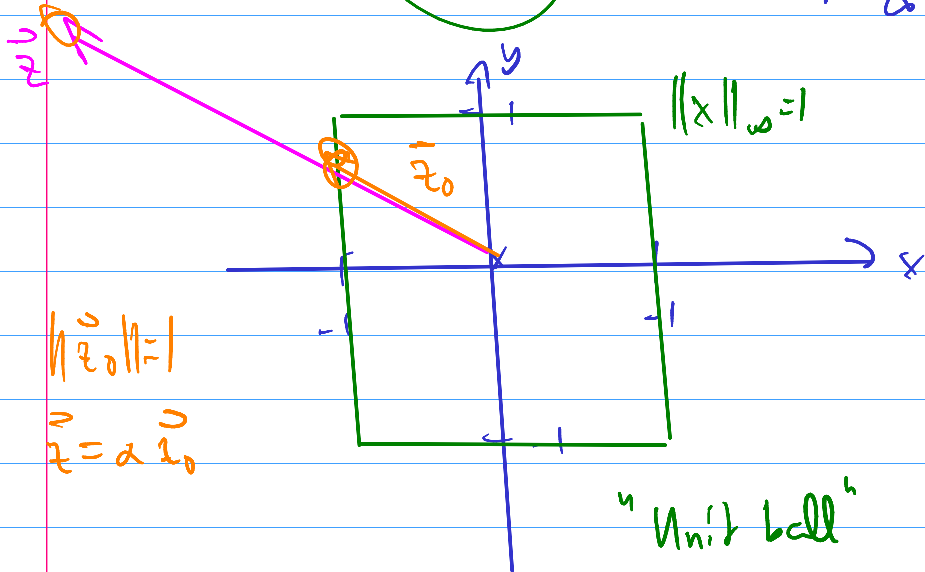
Absolute error $= \|\Delta x\| = \|\vec{x} - \vec{x}_0\| = \|\vec{x}_0 - \vec{x}\|$

Relative error $= \dfrac{\|\Delta x\|}{\|x\|}$



$$\|\vec{x}\|_\infty = 1$$

$$\|\alpha \vec{x}\| = |\alpha|\, \|\vec{x}\|$$

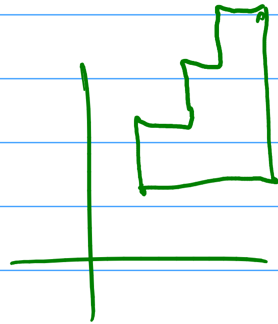


$$\|\vec{z}\| = |\alpha|\, \underbrace{\|\vec{z}_0\|}_{1}$$

$$\|\vec{z}_0\| = 1$$

$$\vec{z} = \alpha \vec{z}_0$$

"Unit ball"



Absolute and Relative Error

- What are the absolute and relative errors in approximating the location of Siebel center (40.114, −88.224) as (40, −88) using the 2-norm?

long: 23

↓ lat: 13

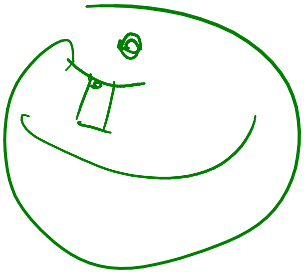
$$\begin{pmatrix}40.114\\-88.224\end{pmatrix} - \begin{pmatrix}40\\-88\end{pmatrix} = \begin{pmatrix}0.114\\-.224\end{pmatrix}$$

Abs. error: $\sqrt{.114^2 + .224^2} = .2513$

Rel. error

$$\left\|\begin{pmatrix}40.114\\-88.224\end{pmatrix}\right\|_2 = 96.97$$

$$\frac{\|\Delta x\|}{\|x_0\|} = .00259$$



Demo: Calculate geographic distances using `tripstance.com`

Matrix Norms

- What norms would we apply to matrices?

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix} \longrightarrow$$

not a matrix norm in our sense

$$\|A\|_F = \sqrt{\sum_{i,j} A_{i,j}^2}$$

Frobenius norm

$$\underbrace{\|A\vec{x}\|}_{\text{known vec norm}} \le \underbrace{\|A\|}_{\text{new mat norm}} \underbrace{\|\vec{x}\|}_{\text{known vec norm}} \rightsquigarrow \frac{\|Ax\|}{\|x\|} \le \|A\|$$

$$\|A\| = \max_{\vec{x} \ne 0} \frac{\|A\vec{x}\|}{\|\vec{x}\|}$$

mat. norm

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max \|Ax\| \cdot \frac{1}{\|x\|}$$

$$= \max_{x \neq 0} \| A \underbrace{\frac{\vec{x}}{\|x\|}}\|$$

$$= \max_{\|y\|=1} \|A\vec{y}\|$$

$$\left\| \begin{pmatrix} 2 & \\ & 3 \end{pmatrix} \right\|_2 = 3 \qquad \left\| \begin{pmatrix} 15 & \\ & 0{,}1 \end{pmatrix} \right\|_\infty$$

$$\rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0{,}5 \\ 0{,}0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Demo: Matrix norms

In-class activity: Matrix norms

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. $\|A\| > 0 \Leftrightarrow A \neq \mathbf{0}$.
2. $\|\gamma A\| = |\gamma| \|A\|$ for all scalars γ.
3. Obeys triangle inequality $\|A + B\| \leqslant \|A\| + \|B\|$

- But also some more properties that stem from our definition: