Overview

Conditioning

Behind the la.solve

↳ fw/subst

↳ LU w/ elim matrices $Ax=b$

Announcements

Examlet 4

More Properties of the Condition Number

$$\text{cond}_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$$

- What is cond(A^{-1})?

$$\text{cond}(A^{-1}) = \|A^{-1}\| \cdot \|(A^{-1})^{-1}\| = \text{cond}(A)$$

- What is the condition number of applying the matrix-vector multiplication $A\boldsymbol{x} = \boldsymbol{b}$? (I.e. now $\boldsymbol{x}$ is the input and $\boldsymbol{b}$ is the output)

$A\vec{x} = \vec{b}$ — $\vec{x}$: input, $\vec{b}$: output

$A x = b$ — x: output, b: input

$B = A^{-1}$

Mult Ax $\equiv$ solve $Bb = x$ for b

$B\vec{b} = \vec{x}$ — $\vec{b}$: out, $\vec{x}$: in

$$\Rightarrow \text{cond}(B) = \text{cond}(A^{-1}) = \text{cond}(A)$$

Matrices with Great Conditioning (Part 1)

- Give an example of a matrix that is *very* well-conditioned. (I.e. has a condition-number that's *good* for computation.) What is the best possible condition number of a matrix?

$$\operatorname{cond}(I) = \|I\| \; \|I\| = 1$$

$$1 = \|I\| = \|AA^{-1}\| \le \|A\| \cdot \|A^{-1}\| = \operatorname{cond}(A)$$

Matrices with Great Conditioning (Part 2)

- What is the 2-norm condition number of an orthogonal matrix A?

$$\text{cond}_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = \|A\|_2 \|A^T\|_2 = 1$$

also orth.

$\rightarrow$ Orthogonal: computationally great

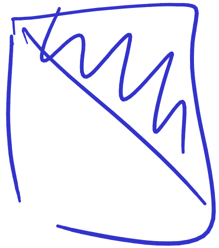
because always well-conditioned

In-class activity: Matrix Conditioning

8 The ‘Undo’ Button for Linear Operations: LU

Solving Systems

- Want methods/algorithms to solve linear systems. Starting small, a kind of system that's easy to solve has a ... matrix.



triangular

Triangular matrices

- Solve

unknown

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

$a_{44} w = b_4 \rightarrow w = \frac{b_4}{a_{44}}$

$a_{33} \cdot z + a_{34} \cdot w = b_3 \rightarrow z = \frac{b_3 - a_{34} \cdot w}{a_{33}}$

$\rightarrow y = \frac{b_2 - a_{24} \cdot w \cdot a_{23} \cdot z}{a_{22}}$

upper tri: "Backward substitution"

lower tri: "Forward subst"

Demo: Coding back-substitution

More General Matrices

- What about non-triangular matrices?

Gaussian elimination

Gaussian Elimination

Demo: Vanilla Gaussian Elimination

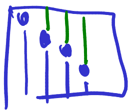
- What do we get by doing Gaussian Elimination?

  Row Echelon Form

- How is that different from being upper triangular?

  because it might have extra zero rows

- What if we do not just eliminate downward but also upward?

   Gauss-Jordan elimination

Elimination Matrices

- What does this matrix do?

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ -\frac{1}{2} & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

$\cdot \left(-\frac{1}{2}\right)$

About Elimination Matrices

- Are elimination matrices invertible?

Yes.

Just flip the sign below the diag

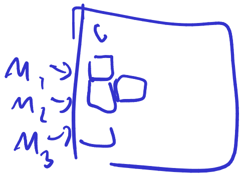
More on Elimination Matrices

Demo: Elimination matrices I

- **Idea:** With enough elimination matrices, we should be able to get a matrix into row echelon form.

- So what do we get from many combined elimination matrices like that?

Demo: Elimination Matrices II



$$\to \underbrace{\cdots M_3 M_2 M_1}_{\text{elim matrices}} A$$

$$\underbrace{M_3 M_2 M_1}\, A = U$$

Products of elimination matrices:

- Just merge their off-diagonal nonzeros
- As long as:
  the nonzeros are in the same column
  or
  are multiplied left-to-right

$$M_3 M_2 M_1 A = U \quad \Big| \; M_3^{-1} \cdot$$

$$M_2 M_1 A = M_3^{-1} U$$

$$\vdots$$

$$A = \underbrace{M_1^{-1} M_2^{-1} M_3^{-1}}_{L} U$$

Summary on Elimination Matrices

- El.matrices with off-diagonal entries in a single column just "merge" when multiplied by one another.
- El.matrices with off-diagonal entries in different columns merge when we multiply (left-column) * (right-column) but not the other way around.
- Inverse: Flip sign below diagonal

LU Factorization

- Can build a factorization from elimination matrices. How?
- Does this help solve $A\boldsymbol{x} = \boldsymbol{b}$?

Demo: LU factorization

In-class activity: LU Factorization

LU: Failure Cases?

- Is LU/Gaussian Elimination bulletproof?

- What can be done to get something *like* an LU factorization?