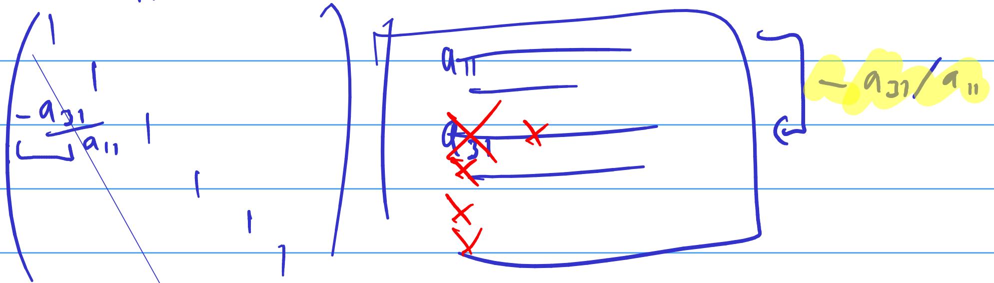
Overview

- Ch
- What it's good for < linear algebra / interpolation

"elimination matrices"



M_4 M_3 M_2

$$\underbrace{M_6}_{\text{third col}}\ \underbrace{M_5 M_4}_{\text{second col}}\ \underbrace{M_3 M_2 M_1}_{\text{for first col}} A = U$$

$$\begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix}$$

$$A = \underbrace{M_1^{-1} \cdots M_3^{-1} M_4^{-1} M_5^{-1} M_6^{-1} U}_{L}$$

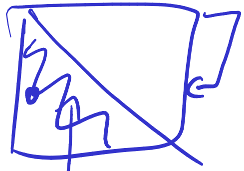
Summary on Elimination Matrices

- El.matrices with off-diagonal entries in a single column just “merge” when multiplied by one another. same column → merge
- El.matrices with off-diagonal entries in different columns merge when we multiply (left-column) * (right-column) but not the other way around.
- Inverse: Flip sign below diagonal

LU Factorization

la.solve

LU:

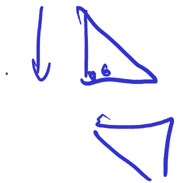


$n^2 \cdot n = O(n^3)$

- Can build a factorization from elimination matrices. How?
- Does this help solve $A\boldsymbol{x} = \boldsymbol{b}$?

$A = LU$

$$L \underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$$



$L\vec{y} = \vec{b}$ ← forward substitution (n^2)

$U\vec{x} = \vec{y}$ ← backward substitution (n^2)

With LU in hand, solving $Ax = b$ costs $O(n^2)$.

Demo: LU factorization

$10^{-6} \leq \kappa \, 10^{-16}$

10^{10}

output cond input

In-class activity: LU Factorization

LU: Failure Cases?

- Is LU/Gaussian Elimination bulletproof?

$$LU = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ l_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} u_{11} & u_{12} \\ & u_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$u_{11} \cdot 1 = 0 \leadsto u_{11} = 0$$

$$l_{21} \cdot \cancel{u_{11}} + 1 \cdot 0 = 2$$

- What can be done to get something *like* an LU factorization?

Swap some rows

permutation matrix

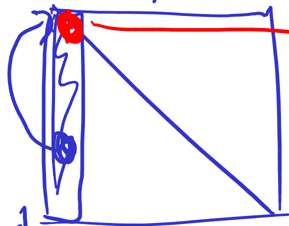
Fixing nonexistence of LU

- How do we capture 'row switches' in a factorization?

$$\begin{pmatrix}1&&&\\&0&1&\\&1&0&\\&&&1\end{pmatrix}\begin{pmatrix}A&A&A&A\\B&B&B&B\\C&C&C&C\\D&D&D&D\end{pmatrix}=\begin{pmatrix}A&A&A&A\\C&C&C&C\\B&B&B&B\\D&D&D&D\end{pmatrix}$$

Mat-mat cost: $O(n)$!

- What does this process look like then?



"pivot"

$$M_3 P_3 M_2 \quad M_1 P_1 A$$

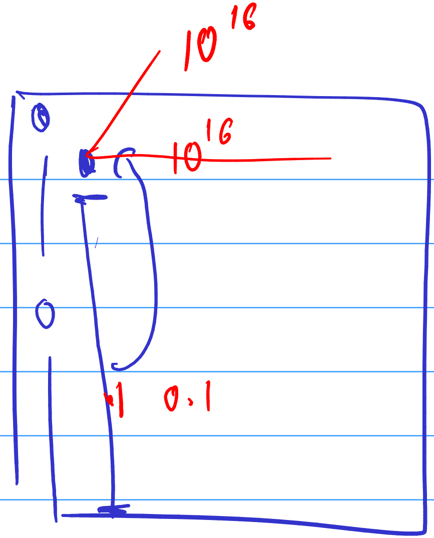
idea: take the biggest entry in the column and swap it onto the diag.

by absolute value

"partial pivoting"

this keeps the factor in the elim. matrix as small as possible

less rounding error



Any permutation matrix (not just row swaps) P can be inverted as P^T

$$M_3 P_3 M_2 P_2 M_1 P_1 A = U$$

$$A = P_1^T M_1^{-1} P_2^T M_2^{-1} P_3^T M_3^{-1} U$$

lower triangular? no!

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$$= \underline{P} \, L \, U \qquad P^T A = LU$$

permutation matrix

Computational Cost

- What is the computational cost of multiplying two $n \times n$ matrices?

- What is the computational cost of carrying out LU factorization on an $n \times n$ matrix?

Demo: Complexity of Mat-Mat multiplication and LU