10 Repeating Linear Operations: Eigenvalues and Steady States

Quiz 22

Overview

Eigenvalues

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- $\boldsymbol{x} \neq \mathbf{0}$ is called an eigenvector of A if there exists a λ so that
$$A\boldsymbol{x} = \lambda\boldsymbol{x}.$$
- In that case, λ is called an eigenvalue.

Finding Eigenvalues

- How do you find eigenvalues?

$$A\vec{x} = \lambda\vec{x}$$

$$\Leftrightarrow (A - \lambda I)\vec{x} = \vec{0}$$

$$\Leftrightarrow A - \lambda I \text{ singular}$$

$$\det(A - \lambda I) = 0$$

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_0 = 0 \quad ?$$

Abel showed a polynomial like this with degree ≥ 5
has no general formula for its roots

$\Rightarrow$ no algorithm w/ finite number of steps

Transforming Eigenvalue Problems

Suppose we know that $Ax = \lambda x$. What are the eigenvalues of these changed matrices?

- Shift. $A \to A - \sigma I$

$(A - \sigma I)\vec{x} = A\vec{x} - \sigma I \vec{x} = \lambda \vec{x} - \sigma \vec{x} = (\lambda - \sigma)\vec{x}$

- Inversion. $A \to A^{-1}$

$A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$

$Ax = \lambda x \rightsquigarrow x = \lambda A^{-1} x$ | A^{-1} $\rightsquigarrow \frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$

- Power. $A \to A^k$

$A^3\vec{x} = AAA\vec{x} = AA\lambda\vec{x} = \lambda^3\vec{x}$

- Inverse. $A \to A^{-1}$

Polynomial $(A^3 + 5A^2 - 7A + 4)\vec{x}$

$= \lambda^3 x + 5\lambda^2 - 7\lambda\vec{x} + 4x$

$= (\lambda^3 + 5\lambda^2 - 7\lambda + 4)\vec{x}$

$A^\sigma x$ ...?

$Ax_1 = \lambda_1 x_1$

$\vdots$

$Ax_n = \lambda_n x_n$

$-1, 2, 5, -7$

Changing Eigenvectors

- Suppose $A\boldsymbol{x} = \lambda \boldsymbol{x}$.
  Can we change the eigenvectors? (but leave the eigenvalues the same)

$\longrightarrow \quad T^{-1}AT \quad \leftarrow$ T invertible matrix

$$T^{-1}AT\vec{y} = \lambda' \vec{y} \quad \Big| \; T\cdot$$

$$AT\vec{y} = \lambda' T\vec{y} \quad \leftarrow \text{Idea: Use } \vec{y} = T^{-1}\vec{x}$$

$$\Leftrightarrow AT(T^{-1}\vec{x}) = \lambda' T(T^{-1}\vec{x})$$

$$\Leftrightarrow A\vec{x} = \lambda \vec{x}$$

$$\lambda' = \lambda$$

The eigenvalues of $T^{-1}AT$ are the same as those of A,

and for each eigenvec. $\vec{x}$ of A, $\vec{y} = T^{-1}\vec{x}$ is an ev of $T^{-1}AT$.

Diagonalizability

- When is a matrix called diagonalizable?

similarity transform $\rightarrow T^{-1}AT = D$ diagonal $\begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$

If you can find a T so that $T^{-1}AT = D$, then A and D are called similar.

Suppose I had all the eigenvectors x_i

$$A\vec{x}_i = \lambda_i \vec{x}_i$$

$$X = \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix}$$

$$AX = X \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{D} \quad | X^{-1}.$$

$$\rightarrow X^{-1}AX = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \leftarrow \text{"diagonalize"}$$

$$\begin{pmatrix}1&&\\&2&\\&&3\end{pmatrix}\begin{pmatrix}\ &\ &\ \\ \ &\ &\ \\ \ &\ &\ \end{pmatrix} =$$

$$\left(\begin{pmatrix}1&&\\&2&\\&&3\end{pmatrix}\right.$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1x_1 & 2x_2 & 3x_3 \end{pmatrix}$$

Are all Matrices Diagonalizable?

- Give characteristic polynomial, eigenvalues, eigenvectors of

$$A = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}.$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ & 1-\lambda \end{pmatrix} = (1-\lambda)^2 \qquad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \rightsquigarrow \begin{matrix} u + v = u \rightsquigarrow v = 0 \\ v = v \end{matrix}$$

all eigenvectors are of the form $\begin{pmatrix} u \\ 0 \end{pmatrix}$

My eigenvector matrix $X = \begin{pmatrix} u_1 & u_2 \\ 0 & 0 \end{pmatrix} \leftarrow$ not invertible

Power Iteration

- What are the eigenvalues of A^{1000}?

$\lambda_1^{1000}, \ldots, \lambda_n^{1000}$

Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
Further assume $\|\boldsymbol{x}_i\| = 1$.

$$\vec{y} = \alpha_1 \vec{x}_1 + \cdots + \alpha_n \vec{x}_n$$

$$A^{1000}\vec{y} = \alpha_1 \lambda_1^{1000}\vec{x}_1 + \cdots + \alpha_n \lambda_n^{1000}\vec{x}_n \quad / \lambda_1^{1000}$$

$$\frac{A^{1000}\vec{y}}{\lambda^{1000}} = \alpha_1 \vec{x}_1 + \alpha_2 \underbrace{\frac{\lambda_2^{1000}}{\lambda_1^{1000}}}_{\left(\frac{\lambda_2}{\lambda_1}\right)^{1000},\ <1} \vec{x}_2 + \cdots + \alpha_n \underbrace{\frac{\lambda_n^{1000}}{\lambda_1^{1000}}}_{\left(\frac{\lambda_n}{\lambda_1}\right)^{1000},\ <1} \vec{x}_n$$

Rayleigh quotient

$$\frac{x_1^T A x_1}{x_1^T x_1} = \frac{x_1^T \lambda_1 x_1}{x_1^T x_1} = \lambda_1$$

Power Iteration: Issues?

- What could go wrong with Power Iteration?

- $\alpha_1 = 0$
- complex-valued eigenvalues/eigenvectors
- $|\lambda_1| = |\lambda_2|$

What about Eigenvalues?

- Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

$\rightarrow$ Rayleigh quotient (see above)

$$\frac{x^T A x}{x^T x}$$

Convergence of Power Iteration

- What can you say about the convergence of the power method? Say $\boldsymbol{v}_1^{(k)}$ is the kth estimate of the eigenvector $\boldsymbol{x}_1$, and

$$e_k = \left\| \boldsymbol{x}_1 - \boldsymbol{v}_1^{(k)} \right\|.$$