Now, let’s study condition number of solving a linear system

$$Ax = b.$$ 

**Input:** \(b\) with error \(\Delta b\),

**Output:** \(x\) with error \(\Delta x\).
Observe \( A(x + \Delta x) = (b + \Delta b) \), so \( A\Delta x = \Delta b \).

\[
\frac{\text{rel err. in output}}{\text{rel err. in input}} = \frac{\|\Delta x\| / \|x\|}{\|\Delta b\| / \|b\|} = \frac{\|\Delta x\| \|b\|}{\|\Delta b\| \|x\|}
\]

\[
= \frac{\|A^{-1}\Delta b\| \|Ax\|}{\|\Delta b\| \|x\|}
\]

\[
\leq \|A^{-1}\| \|A\| \frac{\|\Delta b\| \|x\|}{\|\Delta b\| \|x\|}
\]

\[
= \|A^{-1}\| \|A\|.
\]

So we’ve found an upper bound on the condition number. With a little bit of fiddling, it’s not too hard to find examples that achieve this bound, i.e. that it is tight.
So we’ve found the condition number of linear system solving, also called the condition number of the matrix $A$:

$$\text{cond}(A) = \kappa(A) = \|A\| \|A^{-1}\|.$$  

- $\text{cond}$ is relative to a given norm. So, to be precise, use $\text{cond}_2$ or $\text{cond}_\infty$.

- If $A^{-1}$ does not exist: $\text{cond}(A) = \infty$ by convention.
What is \( \text{cond}(A^{-1}) \)?

\[
\text{cond}(A^{-1}) = \|A\| \cdot \|A^{-1}\| = \text{cond}(A).
\]

What is the condition number of applying the matrix-vector multiplication \( Ax = b \)? (I.e. now \( x \) is the input and \( b \) is the output)

Let \( B = A^{-1} \).

Then computing \( b = Ax \) is equivalent to solving \( Bb = x \).

Solving \( Bb = x \) has condition number \( \text{cond}(B) = \text{cond}(A^{-1}) = \text{cond}(A) \).
So the operation ‘multiply a vector by matrix $A$’ has the same condition number as ‘solve a linear system with matrix $A$’.
Matrices with Great Conditioning (Part 1)

Give an example of a matrix that is very well-conditioned. (I.e. has a condition-number that’s good for computation.)

What is the best possible condition number of a matrix?

Small condition numbers mean not a lot of error amplification. Small condition numbers are good.

The identity matrix $I$ should be well-conditioned:

$$
\|I\| = \max_{\|x\|=1} \|Ix\| = \max_{\|x\|=1} \|x\| = 1.
$$

It turns out that this is the smallest possible condition number:

$$
1 = \|I\| = \|A \cdot A^{-1}\| \leq \|A\| \cdot \|A^{-1}\| = \kappa(A).
$$
Both of these are true for any norm $\|\cdot\|$. 
What is the 2-norm condition number of an orthogonal matrix $A$?

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = \|A\|_2 \|A^T\|_2 = 1.$$ 

That means orthogonal matrices have optimal conditioning. They’re very well-behaved in computation.