Overview

SVD

\[ \langle \mathbf{sa} \rangle \| \langle \mathbf{a} \rangle \|^2 \]

\[ \mathbf{c} \to \mathbf{CRA} \]

Interpolation
Fitting a Model to Data

How can I fit a model to measurements? E.g.:

\[ p(x) = \alpha_0 + \alpha_1 \times f(x) + \epsilon \]

Find \( \mathbf{x} \) s.t. \( \mathbf{V} \left( \mathbf{x} - \mathbf{y} \right) \mathbf{l} \) is minimal
\[ V \xi \nRightarrow b \]

"least-squares equal"

i.e., minimize the 2-norm of the residual
**Demo:** Data Fitting using Least Squares
Meaning of the Singular Values

What do the singular values mean? (in particular the first/largest one)

\[ A = U \Sigma V^T \]

\[ \|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 \]

\[ = \max_{\|x\|_2 = 1} \|U \Sigma V^T x\|_2 \]

\[ = \max_{\|x\|_2 = 1} \| \Sigma V^T x \|_2 \]

\[ \rightarrow \|x \|_2 = 1 \]

same
\[
\begin{align*}
\text{L} & = \max_{V \in \mathcal{M}} \|CV^*\|_2 \\
\Rightarrow \|V^*\|_2 & = 1 \\
\|V^*\|_2 & = 1 \\
\|V^*\|_2 & = \sigma_1 \\
\|Qx\|_2 & = (Qx)^T (Qx) = x^T Q^T Q x \\
\|Qx\|_2 & = \frac{x^T Q^T Q x}{\|x\|_2^2}
\end{align*}
\]
How would you compute a 2-norm condition number?

\[ \text{cond}_2(A) = \| A \|_2 \| A^{-1} \|_2 = \sigma_1 \cdot \frac{1}{\sigma_n} \]

\[ A^{-1} = V \left( \begin{array}{c} \frac{1}{\sigma_1} \\ \vdots \\ \frac{1}{\sigma_n} \end{array} \right) \sigma \]
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SVD as Sum of Outer Products

What’s another way of writing the SVD?

\[ A = U \Sigma V^T = \left( \begin{array}{ccc} \sigma_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma_n \end{array} \right) \left( \begin{array}{c} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_m^T \end{array} \right) \left( \begin{array}{c} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{array} \right)^T \]

\[ = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_n \mathbf{u}_n \mathbf{v}_n^T \]
\[ A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T \]

\[ A_z = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T \]

\[ 3(n+1) + 3 \]

\[ A_3 \approx A \]
$F = 100 \left( \begin{array}{c} 100 \\ - \end{array} \right) \begin{pmatrix} \delta_1 & \ldots & \delta_{5000} \end{pmatrix}$, big

$\sum_{i=1}^{5000} \left( \begin{array}{c} 100 \\ \delta_i \end{array} \right)$
Low-Rank Approximation (I)

What is the rank of $\sigma_1 u_1 v_1^T$?

What is the rank of $\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$?

**Demo:** Image Compression
Low-Rank Approximation

What can we say about the low-rank approximation

\[ A_k = \sigma_1 u_1 v_1^T + \sigma_k u_k v_k^T \]

to

\[ A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T \]

For simplicity, assume \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0 \).

\[ \min_{\text{rank } B = k} \| A - B \|_2 = \| A - A_k \|_F \]

( Eckart-Young-Mirsky theorem)

\[ \min_{\text{rank } B = k} \| A - B \|_F = \| A - A_k \|_F \]
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