Overview:

- Taylor (demo/ interactive) (derivative → function)
- applying polynomial approximations (compute π)
- interpolation (point values → function)

Reconstructing a Function From Derivatives

Found: *Taylor series approximation*.

$$f(0+x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots$$

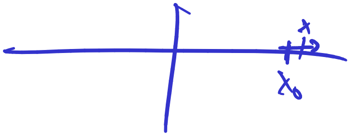
The general Taylor expansion with center $x_0 = 0$ is

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

Demo: Polynomial Approximation with Derivatives (Part I)

Shifting the Expansion Center

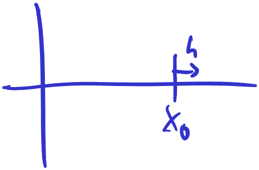
- Can you do this at points other than the origin?



$$f\big(x_0 + \underline{(x - x_0)}\big) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i$$

$h = x - x_0$

$$f(x_0 + h) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!} h^i$$



Errors in Taylor Approximation (I)

- Can't sum infinitely many terms. Have to *truncate*. How big of an error does this cause?

  Demo: Polynomial Approximation with Derivatives (Part II)

$$\left| f(x_0+h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i \right| \le \underline{C} \cdot h^{n+1}$$

$$= O(h^{n+1})$$

Making Predictions with Taylor Truncation Error

- Suppose you expand $\sqrt{x-10}$ in a Taylor polynomial of degree 3 about the center $x_0 = 12$. For $h_1 = 0.05$, you find that the Taylor truncation error is about 10^{-4}.

  What is the Taylor truncation error for $h_2 = 0.025$?

$$\text{Error}(h) = \left| f - \sum^{n} \text{Taylor} \right| \approx C \cdot h^{n+1}$$

$$\text{Error}(h_1 = 0.05) \approx 10^{-4} \approx C \cdot h_1^4$$

$$h_2 = \frac{h_1}{2}$$

$$\text{Error}(h_2 = 0.025) \approx C \cdot h_2^4 = C \cdot \left(\frac{h_1}{2}\right)^4 = \underbrace{C \cdot h_1^4}_{= \text{Error}(h_1)} \cdot \left(\frac{1}{2}\right)^4$$

Demo: Polynomial Approximation with Derivatives (Part III)

Taylor Remainders: the Full Truth

Let $f : \mathbb{R} \to \mathbb{R}$ be $(n+1)$-times differentiable on the interval (x_0, x) with $f^{(n)}$ continuous on $[x_0, x]$. Then there exists a $\xi \in (x_0, x)$ so that

$$f(x_0+h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}}_{\text{"}C\text{"}} \cdot (\xi - x_0)^{n+1}$$

and since $|\xi - x_0| \leqslant h$

$$\left| f(x_0+h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i \right| \leqslant \underbrace{\frac{\left|f^{(n+1)}(\xi)\right|}{(n+1)!}}_{\text{"}C\text{"}} \cdot h^{n+1}.$$

Proof of Taylor Remainder Theorem

- Intuitively the error of an approximation that takes into account n derivatives should be proportional to the maximum value of the $(n+1)$th one...

In-class activity: Taylor series

Using Polynomial Approximation

- Suppose we can approximate a function as a polynomial:

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

How is that useful? Say, if I wanted the integral of f?

Demo: Computing π with Taylor