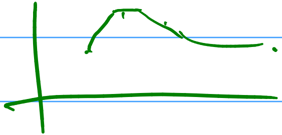
Overview

- Interpolation



  - Error
  - Application
- Monte Carlo

Exam overbooking
RELATE
maintenance
recruiting
HW2

Vandermonde Linear Systems

Polynomial interpolation is a critical component in many numerical models.

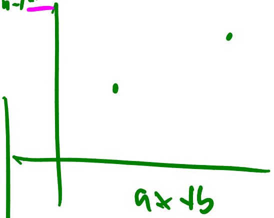
$$(x_1, y_1) \quad \dots \quad (x_n, y_n)$$

$$p(x) = a_0 \underline{1} + a_1 \underline{x} + a_2 \underline{x^2} + \dots + a_{n-1} \underline{x^{n-1}}$$

$$p(x_i) = y_i$$

$$\text{points} \downarrow \begin{pmatrix} 1 & x_1 & & x_1^{n-1} \\ 1 & & & \\ \vdots & & & \\ 1 & x_n & & x_n^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \\ \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ \\ \\ y_n \end{pmatrix}$$

$\longrightarrow$ # functions



Demo: Polynomial Approximation with Point Values

Error in Interpolation

How did the interpolation error behave in the demo?

To fix notation: f is the function we're interpolating. $\tilde{f}$ is the interpolant that obeys $\tilde{f}(x_i) = f(x_i)$ for $x_i = x_1 < \ldots < x_n$. $h = x_n - x_1$ is the interval length.

$O(h^n)$

What is the error *at* the interpolation nodes?

0

Care to make an unfounded prediction? What will you call it?

convergence of order n

n-th order convergence

Proof Intuition for Interpolation Error Bound

Let us consider an interpolant $\tilde{f}$ based on $n = 2$ points so

$$\tilde{f}(x_1) = f(x_1) \quad \text{and} \quad \tilde{f}(x_2) = f(x_2).$$

Prove the interpolation error bound in this case.

$E(x) = f(x) - p(x)$

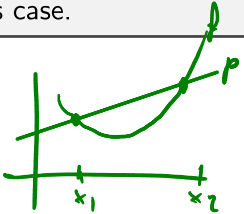
$E(x_1) = 0 \qquad E(x_2) = 0$

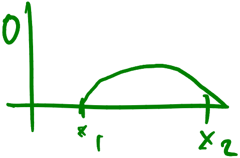
$\int_{x_1}^{x_2} E'(x)\,dx = E(x_2) - E(x_1) = 0$

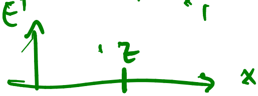
What does that say about E'?

Case I: $E'(x) = 0$

Case II: It has a non-zero value







There exists a z so that $E'(z) = 0$

$$E(x) = \overbrace{E(x_1)}^{0} + \int_{x_1}^{x} E'(w_0)\, dw_0$$

$$= \int_{x_1}^{x} \cancel{E'(z)} + \int_{z}^{w_0} E''(w_1)\, dw_1\, dw_0$$

$$= \int_{x_1}^{x} \int_{z}^{w_0} E''(w_1)\, dw_1\, dw_0$$

$$\left[E(x) = f(x) - \overbrace{ax+b}^{p(x)} \right] \quad E''(x) = f''(x)$$

$$C = \max_{x \in [x_1, x_2]} |f''(x)|$$

$$\leq \int_{x_1}^{x} \int_{z}^{w_0} C\, dw_1\, dw_2$$

$$\leq C \cdot \frac{h^2}{2} = \max |f''(x)| \cdot \frac{h^2}{2}$$

Making Use of Interpolants

Suppose we can approximate a function as a polynomial:

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

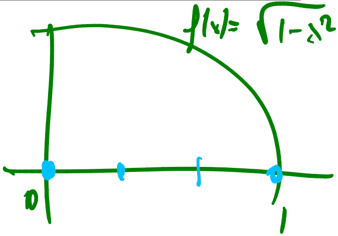
How is that useful? E.g. what if we want the integral of f?

Step 1: Interpolate, get polynomial $p(x)$.

Step 2: Claim that

$$\int_0^1 f(x)\,dx \approx \int_0^1 p(x)\,dx$$

Step 3: $4 \cdot \text{result} \approx \pi$



Demo: Computing π with Interpolation

$$p(x) = \underline{\qquad} \cdot \underbrace{1}_{\varphi_0(x)=1} + \underline{\qquad} \cdot \underbrace{x}_{\varphi_1(x)=x} + \underline{\qquad} \cdot \underbrace{x^2}_{\varphi_2(x)=x^2}$$

$$\varphi_i(x) = x^i$$

$$\varphi_i(x) = \sin(i\,x)$$

$\varphi_0(x) =$ 

$\varphi_1(x) =$ 

More General Functions

Is this technique limited to the monomials $\{1, x, x^2, x^3, \dots\}$?

$$V = \begin{array}{c} \cancel{1 \quad x \quad x^2} \\ \begin{pmatrix} \varphi_0(x_1) & \cdots & \varphi_{n-1}(x_1) \\ \vdots & & \\ \varphi_0(x_n) & \cdots & \varphi_{n-1}(x_n) \end{pmatrix} \end{array}$$

Interpolation with General Sets of Functions

For a general set of functions $\{\varphi_1, \ldots, \varphi_n\}$, solve the linear system with the generalized Vandermonde matrix for the coefficients $(a_1, \ldots, a_n)$:

$$\underbrace{\begin{pmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(x_n) & \varphi_2(x_n) & \cdots & \varphi_n(x_n) \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}}_{\boldsymbol{a}} = \underbrace{\begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}}_{\boldsymbol{f}}.$$

Given those coefficients, what is the interpolant $\tilde{f}$ satisfying $\tilde{f}(x_i) = f(x_i)$?

$$\tilde{f}(x) = a_1 \cdot \varphi_0(x) + a_1 \cdot \varphi_1(x) + \cdots + a_n \varphi_{n-1}(x)$$

In-class activity: Interpolation

Outline

Randomness: Why?

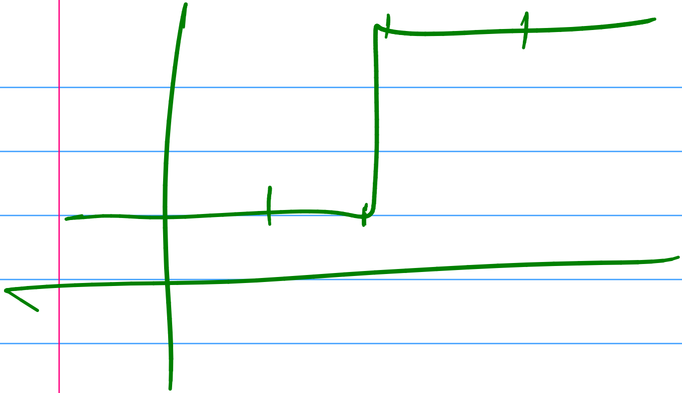
What types of problems can we solve with the help of random numbers?

We can compute (potentially) *complicated averages*.

- Where does 'the average' web surfer end up? (PageRank)
- How much is my stock portfolio/option going to be worth?
- How will my robot behave if there is measurement error?

4 2 3 1 6 1 2

12:34
15
now



Random Variables

What is a random variable?

A random variable X is a function that depends on 'the (random) state of the world'.

Example: X could be

- 'how much rain tomorrow?', or
- 'will my buttered bread land face-down?'

Idea: Since I don't know the entire state of the world (i.e. all the influencing factors), I can't know the value of X.

$\rightarrow$ Next best thing: Say something about the *average* case.

To do that, I need to know how likely each individual value of X is. I need a probability distribution.

Probability Distributions

What kinds of probability distributions are there?

Demo: Plotting Distributions with Histograms