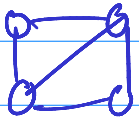
Laplacian Systems



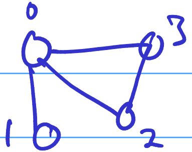
Adjacency matrix A

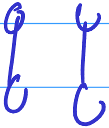
G - unweighted

$A_{ij} \in \{0, 1\}$

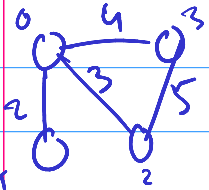
$L = D - A$ where D_{ii} = degree of vertex i

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & & \\ -1 & & 2 & -1 \\ -1 & & -1 & 2 \end{bmatrix}$$





$$\left[\begin{array}{cc|cc} 1 & -1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & 1 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & -1 \\ & & -1 & 1 \end{array}\right] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} z_1 \\ 0 \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & & \\ 3 & & 0 & 5 \\ 4 & & 5 & 0 \end{bmatrix}$$

$$L = D - A$$

$$D_{ii} = \sum_{j=1}^{n} A_{ij}$$

$$L = \begin{bmatrix} 9 & -2 & -3 & -4 \\ -2 & 2 & & \\ -3 & & 8 & -5 \\ -4 & & -5 & 9 \end{bmatrix}$$

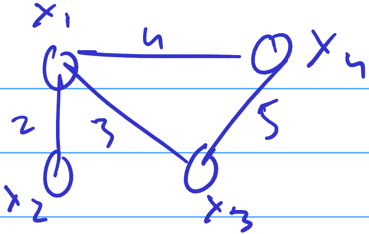
$$Lx = b$$

$$b_1 = 9x_1 - 2x_2 - 3x_3 - 4x_4$$

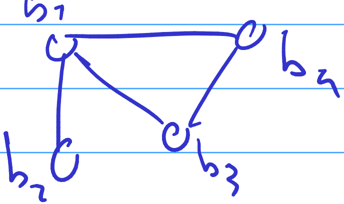
$$b_1 = \sum_j A_{ij}(x_1 - x_j)$$

$$= (2x_1 - 2x_3) + (3x_1 - 3x_3) +$$

$$(4x_1 - 4x_4)$$



$b = Lx_1$



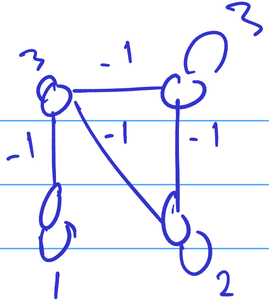
$0 = Lx_1$

$b_1 = 9x_1 - (2x_2 + 3x_3 + 4x_4)$

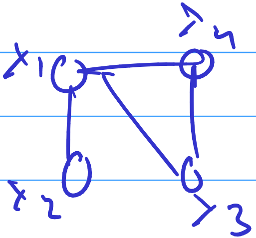
$0 = 9x_1 - (2x_2 + 3x_3 + 4x_4)$

$9x_1 = 2x_2 + 3x_3 + 4x_4$

$8x_3 = 3x_1 + 5x_4$



$\Rightarrow$ adjacency matrix $= L$



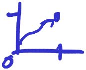
$$3x_1 = x_2 + x_3 + x_4$$

$$x_1 = \frac{x_2 + x_3 + x_4}{3}$$

Outline

Norms

What's a norm?

a measure of magnitude  

"absolute value for vectors"

Define norm.

$$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$$

$$\|\gamma \cdot x\| = |\gamma| \cdot \|x\|$$

if $\|x\| = 0$ then $x = 0$

$$\|x + y\| \leq \|x\| + \|y\|$$

$$\|x\| \geq 0$$

given x

$$\|\alpha \cdot x\| = 1$$

$$\alpha = \frac{1}{\|x\|}$$

$$\left\|\frac{1}{\|x\|} x\right\| = 1$$

Examples of Norms

What are some examples of norms?

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

$$\left\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|_1 = (1 + 2)^1 = 3$$

$$\left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\|_1 = (1 \;\; 1 \;\; 2)^1 = 3$$

$$\left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\|_2 = \sqrt{\left(1^2 + (-2)^2\right)} = \sqrt{5}$$

$$\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\|_p = \left(1^p + 1^p + 1^p + 1^p\right)^{\frac{1}{p}} = 4^{\frac{1}{p}}$$

$$\left\| \begin{pmatrix} 1 \\ 10 \end{pmatrix} \right\|_1 = 11$$

$$\left\| \begin{pmatrix} 1 \\ 10 \end{pmatrix} \right\|_2 = \sqrt{101}$$

$$\sqrt{10^2 + 1^2}$$

$$\left\| \begin{pmatrix} 1 \\ 10 \end{pmatrix} \right\|_\infty = \cancel{\left(1^\infty + 10^\infty\right)^{\frac{1}{\infty}}} = 10$$

$$\|v\|_\infty = \max_i (|v_i|)$$

Demo: Vector norms

Norms and Errors

If we're computing a vector result, the error is a vector.
That's not a very useful answer to 'how big is the error'.
What can we do?

$$\|\text{correct} - \text{computed}\| = \text{absolute error}$$

$$\|\text{correct}\| - \|\text{computed}\| = ?$$

$$\frac{\|\text{correct} - \text{computed}\|}{\|\text{correct}\|} = \text{relative error}$$

Absolute and Relative Error

What are the absolute and relative errors in approximating the location of Siebel center $(40.114, -88.224)$ as $(40, -88)$ using the 2-norm?

abs err. $\left\| \begin{pmatrix} 40.114 \\ -88.224 \end{pmatrix} - \begin{pmatrix} 40 \\ -88 \end{pmatrix} \right\|_2 = \left\| \begin{matrix} .114 \\ -.224 \end{matrix} \right\|_2$

$$= \left(.114^2 + .224^2\right)^{1/2}$$

rel err $= \dfrac{\left(.114^2 + .224^2\right)^{1/2}}{\left\| \begin{pmatrix} 40.114 \\ -88.224 \end{pmatrix} \right\|_2}$

Demo: Calculate geographic distances using `tripstance.com`

Matrix Norms

What norms would we apply to matrices?

$$\|\mathrm{vec}(A)\|_2 = \left(\sum_i \sum_j A_{ij}^2\right)^{1/2} = \|A\|_F$$

$$\neq \|A\|_2$$

Frobenius norm $\|\cdot\|_F$ is entrywise

induced matrix norms, e.g. $\|\cdot\|_p$

$$\|A\|_p = \max_{\|x\|_p = 1} \|A \cdot x\|_p = \max_{x \in \mathbb{R}^n} \frac{\|Ax\|_p}{\|x\|_p}$$

Demo: Matrix norms

In-class activity: Matrix norms

$$\begin{pmatrix} 3 & 0 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a \\ 4a+4b \end{pmatrix}$$

$$\left\| \begin{pmatrix} 3a \\ 4a+4b \end{pmatrix} \right\|_1 = 7|a| + 4|b|$$

$$|a| + |b| = 1$$

$$\left\| \begin{pmatrix} 3 & 0 \\ 4 & 4 \end{pmatrix} \right\|_1 = 7$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left\| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right\|_F = \sqrt{2^2 + 1^2 + 1^2 + 2^2} = \sqrt{10}$$

$$\left\| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right\|_2 = \max_{x \in R^2} \left\| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \frac{x}{\|x\|_2} \right\|_2$$

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Bigg| \quad Ax_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\|Ax_1\|_2 = \sqrt{2^2 + 1} = \sqrt{5}$$

max $\|y\|_2$, $y = Ax$ for $\|x\|_2 = 1$

for any z, $\|Az\|_2 \leq \|A\|_2 \cdot \|z\|$

submultiplicativity

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. $\|A\| > 0 \Leftrightarrow A \neq \mathbf{0}$.
2. $\|\gamma A\| = |\gamma| \, \|A\|$ for all scalars γ.
3. Obeys triangle inequality $\|A + B\| \leqslant \|A\| + \|B\|$

But also some more properties that stem from our definition: