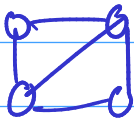


Laplacian Systems



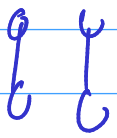
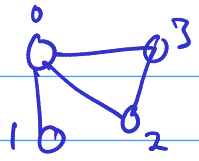
Adjacency matrix A

G -unweighted

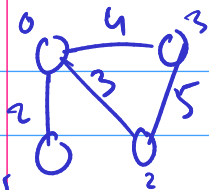
$$A_{ij} \in \{0, 1\}$$

$$L = D - A \quad \text{where } D_{ii} = \text{degree of vertex } i$$

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & & \\ -1 & & 2 & -1 \\ -1 & & -1 & 2 \end{bmatrix}$$



$$\left[\begin{array}{ccc|ccc} 1 & r_1 & & & & 0 \\ -1 & & 1 & & & \\ \hline 0 & & & 1 & -1 & \\ & & & -1 & 1 & \end{array} \right] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} z_1 \\ 0 \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & & \\ 3 & & 0 & 5 \\ 4 & & 5 & 0 \end{bmatrix}$$

$$L = D - A$$

$$D_{ii} = \sum_{j=1}^n A_{ij}$$

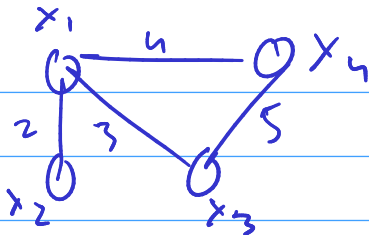
$$L = \begin{bmatrix} 9 & -2 & -3 & -4 \\ -2 & 2 & & \\ -3 & & 8 & -5 \\ -4 & & -5 & 9 \end{bmatrix}$$

$$Lx = b$$

$$b_1 = 9x_1 - 2x_2 - 3x_3 - 4x_4$$

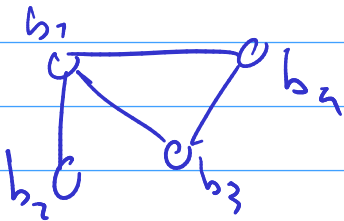
$$b_1 = \sum_j A_{1j} (x_1 - x_j) \leftarrow$$

$$= (2x_1 - 2x_3) + (3x_1 - 3x_3) + (4x_1 - 4x_4)$$



$$b = Lx_1$$

$$0 = Lx_1$$

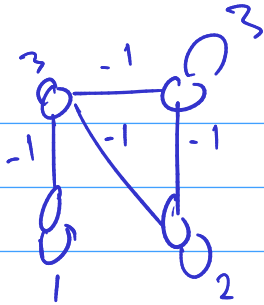


$$b_1 = 9x_1 - (2x_2 + 3x_3 + 4x_4)$$

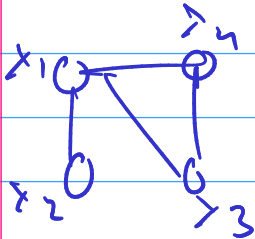
$$0 = 9x_1 - (2x_2 + 3x_3 + 4x_4)$$

$$9x_1 = 2x_2 + 3x_3 + 4x_4$$

$$8x_3 = 3x_1 + 5x_4$$



\Rightarrow adjacency
matrix = L



$$3x_1 = x_2 + x_3 + x_4$$

$$x_1 = \frac{x_2 + x_3 + x_4}{3}$$

Outline

Python, Numpy, and Matplotlib
Making Models with Polynomials
Making Models with Monte Carlo

Error, Accuracy and Convergence
Floating Point

Modeling the World with Arrays

- The World in a Vector

- What can Matrices Do?

- Graphs

- Sparsity

Norms and Errors

The 'Undo' Button for Linear Operations: LU

LU: Applications

- Linear Algebra Applications

- Interpolation

Repeating Linear Operations:
Eigenvalues and Steady States

Eigenvalues: Applications

Approximate Undo: SVD and Least Squares

SVD: Applications

- Solving Funny-Shaped Linear Systems

- Data Fitting

- Norms and Condition

- Numbers

- Low-Rank Approximation

Iteration and Convergence

Solving One Equation


Solving Many Equations

Finding the Best: Optimization in 1D

Optimization in n Dimensions

Norms

What's a norm?

a measure of magnitude
"absolute value for vectors" 

Define norm.

$$\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}_+$$

$$\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$$

if $\|x\| = 0$ then $x = 0$

$$\|x + y\| \leq \|x\| + \|y\|$$

$$\|x\| \geq 0$$

given x

$$\|x \cdot x\| = 1$$

$$\alpha = \frac{1}{\|x\|}$$

$$\left\| \frac{1}{\|x\|} x \right\| = 1$$

Examples of Norms

What are some examples of norms?

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

$$\left\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|_1 = (1 + 2)^1 = 3$$

$$\left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\|_1 = (1 + 2)^1 = 3$$

$$\left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\|_2 = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

$$\left\| \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\|_p = \left(1^p + 1^p + 1^p + 1^p \right)^{\frac{1}{p}} = 4^{\frac{1}{p}}$$

$$\left\| \begin{pmatrix} 1 \\ 10 \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} 1 \\ 10 \end{pmatrix} \right\|_2 = \frac{\sqrt{101}}{\sqrt{10^2 + 1^2}}$$

$$\left\| \begin{pmatrix} 1 \\ r_0 \end{pmatrix} \right\|_{\infty} = \left(1^{\infty} + |r_0|^{\infty} \right)^{\frac{1}{\infty}} = 1$$

$$\|v\|_{\infty} = \max_i (|v_i|)$$

Demo: Vector norms

Norms and Errors

If we're computing a vector result, the error is a vector.
That's not a very useful answer to 'how big is the error'.
What can we do?

$$\| \text{correct} - \text{computed} \| = \text{absolute error}$$

$$\| \text{correct} \| - \| \text{computed} \| = ?$$

$$\frac{\| \text{correct} - \text{computed} \|}{\| \text{correct} \|} = \text{relative error}$$

Absolute and Relative Error

What are the absolute and relative errors in approximating the location of Siebel center $(40.114, -88.224)$ as $(40, -88)$ using the 2-norm?

$$\text{abs err.} \quad \left\| \begin{pmatrix} 40.114 \\ -88.224 \end{pmatrix} - \begin{pmatrix} 40 \\ -88 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} .114 \\ -.224 \end{pmatrix} \right\|_2$$

$$= \left(.114^2 + .224^2 \right)^{1/2}$$

$$\text{rel err} = \frac{\left(.114^2 + .224^2 \right)^{1/2}}{\left\| \begin{pmatrix} 40.114 \\ -88.224 \end{pmatrix} \right\|_2}$$

Demo: Calculate geographic distances using tripstance.com

Matrix Norms

What norms would we apply to matrices?

$$\| \text{vec}(A) \|_2 = \left(\sum_i \sum_j A_{ij}^2 \right)^{1/2} = \|A\|_F$$

$$\neq \|A\|_2$$

Frobenius norm $\|\cdot\|_F$ is entrywise induced matrix norms, e.g. $\|\cdot\|_p$

$$\|A\|_p = \max_{\|x\|_p=1} \|A \cdot x\|_p = \max_{x \in \mathbb{R}^n} \frac{\|A \cdot x\|_p}{\|x\|_p}$$

Demo: Matrix norms

In-class activity: Matrix norms

$$\begin{pmatrix} 3 & 0 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a \\ 4a + 4b \end{pmatrix}$$

$$\left\| \begin{pmatrix} 3a \\ 4a + 4b \end{pmatrix} \right\|_1 = 7|a| + 4|b|$$

$$\left\| \begin{pmatrix} 3 & 0 \\ 4 & 4 \end{pmatrix} \right\|_1 = 7 \quad \begin{matrix} |a| + |b| = 1 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{matrix}$$

$$\left\| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right\|_F = \sqrt{2^2 + 1^2 + 1^2 + 2^2} = \sqrt{10}$$

$$\left\| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right\|_2 = \max_{x \in \mathbb{R}^2} \left\| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \frac{x}{\|x\|_2} \right\|_2$$

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \left| \quad Ax_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right.$$
$$\|Ax_1\|_2 = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\max \|y\|_2, \quad y = Ax \quad \text{for } \|x\|_2 = 1$$

for any z , $\|Az\|_2 \leq \|A\|_2 \cdot \|z\|_2$
submultiplicativity

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. $\|A\| > 0 \Leftrightarrow A \neq \mathbf{0}$.
2. $\|\gamma A\| = |\gamma| \|A\|$ for all scalars γ .
3. Obeys triangle inequality $\|A + B\| \leq \|A\| + \|B\|$

But also some more properties that stem from our definition: