

# Matrix Conditioning


$$A \in \mathbb{R}^{n \times n}$$

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2$$

$$Ax = b$$

Solving for  $x$  has condition number  $\kappa(A)$

Multiplying  $x$  by  $A$  has  
condition number  $\kappa(A)$

$$\kappa(A) = \kappa(A^{-1}) = \|A\|_2 \cdot \|A^{-1}\|_2$$


The same problem as solving  
for  $b$  in  $A^{-1}b = x$

## More Properties of the Condition Number

What is  $\text{cond}(A^{-1})$ ?

$$\kappa(A^{-1}) = \|A^{-1}\| \|A\| = \kappa(A)$$

What is the condition number of applying the matrix-vector multiplication  $Ax = b$ ? (i.e. now  $x$  is the input and  $b$  is the output)

$$\kappa(A)$$

conditioning =  $\frac{\text{rel err in output}}{\text{rel err in input}}$

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|$$

$\|A\|$  = maximum growth in obser

$\|A^{-1}\|$  = minimum growth in input

## Matrices with Great Conditioning (Part 1)

Give an example of a matrix that is *very* well-conditioned.  
(I.e. has a condition-number that's *good* for computation.)  
What is the best possible condition number of a matrix?

$$\kappa(I) = 1$$

$$\kappa\left(\begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix}\right) = 1$$

Or the general matrix  $Q$

$$Q^T Q = I \Rightarrow Q^T = Q^{-1}$$

$$\|Q\|_2 = \sqrt{\langle x Q^T, Q x \rangle} = \sqrt{\langle x, x \rangle} = 1$$

$\max_{\|x\|_2=1}$

$$\kappa(Q) = \|Q\|_2 \cdot \|Q^T\|_2$$

$$= 1 \cdot \|Q^T\|_2$$

$$= 1 \cdot 1 = 1$$

$$\|A \cdot A^{-1}\| = 1$$



$$\|A \cdot A^{-1}\| \leq \|A\| \cdot \|A^{-1}\|$$

$$\|x \cdot y\| \leq \|x\| \cdot \|y\|$$

Substit

$$\kappa(A) = \|A\| \cdot \|A^{-1}\| \geq 1$$

for any  $p$ -norm

## Matrices with Great Conditioning (Part 2)

What is the 2-norm condition number of an orthogonal matrix  $A$ ?

$$\|A\|_2 = 1$$

## In-class activity: Matrix Conditioning



# Outline

Python, Numpy, and Matplotlib  
Making Models with Polynomials  
Making Models with Monte Carlo  
Error, Accuracy and Convergence  
Floating Point  
Modeling the World with Arrays  
    The World in a Vector  
    What can Matrices Do?  
    Graphs  
    Sparsity  
Norms and Errors  
**The 'Undo' Button for Linear Operations: LU**  
Repeating Linear Operations:  
Eigenvalues and Steady States  
Eigenvalues: Applications

Approximate Undo: SVD and Least Squares  
SVD: Applications  
    Solving Funny-Shaped Linear Systems  
    Data Fitting  
    Norms and Condition Numbers  
    Low-Rank Approximation  
Interpolation  
Iteration and Convergence  
Solving One Equation  
Solving Many Equations  
Finding the Best: Optimization in 1D  
Optimization in  $n$  Dimensions

# Solving Systems of Equations

Want methods/algorithms to solve linear systems. Starting small, a kind of system that's easy to solve has a ... matrix.

The diagram shows a system of two linear equations in two variables, enclosed in a hand-drawn box. The equations are  $a_{11}x_1 + a_{21}x_2 = b_1$  and  $a_{12}x_1 + a_{22}x_2 = b_2$ . The terms  $a_{11}$ ,  $a_{21}$ ,  $b_1$ ,  $a_{12}$ ,  $a_{22}$ , and  $b_2$  are underlined. A large 'X' is drawn over the entire system. The word "given" is written below the equations. To the right of the box, the equations are solved for  $x_2$  and  $x_1$ .

$$a_{11}x_1 + a_{21}x_2 = b_1$$
$$a_{12}x_1 + a_{22}x_2 = b_2$$

given

$$x_2 = \frac{b_2}{a_{22}}$$
$$a_{11}x_1 = b_1 - a_{21} \left( \frac{b_2}{a_{22}} \right)$$

# Triangular Matrices

Solve

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

$$x_4 = b_4 / a_{44}$$
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 - a_{14} \cdot x_4 \\ b_2 - a_{24} \cdot x_4 \\ b_3 - a_{34} \cdot x_4 \end{pmatrix}$$

**Demo:** Back-substitution

**In-class activity:** Forward-substitution

$$\begin{pmatrix} a_{11} \\ b_{21} & a_{12} \\ b_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

## General Matrices

What about non-triangular matrices?

Gaussian Elimination =  $LU$

Given  $n \times n$  matrix  $A$ , obtain lower triangular matrix  $L$  and upper triangular matrix  $U$  such that  $A = LU$ .

Is there some redundancy in this representation?

$$n(n+1)/2 = \sum_{i=1}^n i = \frac{n}{2} \cdot (n+1)$$

$$1 + 2 + 3 + 4 + 5 + 6 = 3 \cdot 7$$

diagonal of  $L$  is 1  
"unit" diagonal

## Using LU Decomposition to Solve Linear Systems

Given  $A = LU$ , how do we solve  $Ax = b$ ?

$$L \cdot U \cdot x = b$$

$\underbrace{\hspace{2cm}}$   
 $y$

$$L \cdot y = b \quad \leftarrow \text{solve by fwd. subs.}$$

$$U \cdot x = y \quad \leftarrow \text{solve for } x \text{ by back. subs.}$$

$$x = U^{-1} L^{-1} b$$

$\uparrow \quad \uparrow$   
—————

do not compute  
but apply

## 2-by-2 LU Factorization (Gaussian Elimination)

Lets consider an example for  $n = 2$ .

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & a_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} \\ 0 \end{bmatrix} = \begin{bmatrix} u_{11} \\ l_{21}u_{11} \end{bmatrix}$$

$$a_{21} = l_{21} u_{11} \Rightarrow a_{21} \cdot u_{11}^{-1} \\ = a_{21} / u_{11}$$

$$a_{22} = [l_{21} \ 1] \cdot \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}$$

$$a_{22} = l_{21} \cdot u_{12} + u_{22}$$

$$u_{22} = a_{22} - \underbrace{l_{21} \cdot u_{12}}_{\text{Schritt complement}}$$



## General LU Factorization (Gaussian Elimination)

$$A = \begin{bmatrix} a_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{l}_{21} & L_{22} \end{bmatrix} \cdot \begin{bmatrix} u_{11} & \mathbf{u}_{12} \\ 0 & U_{22} \end{bmatrix}$$

Handwritten diagram illustrating the partitioning of matrix  $A$  into blocks. A large bracket on the left indicates the full matrix  $A$ . Inside, a vertical line separates the first column from the rest of the matrix. The top-left element is labeled  $a_{11}$ . The top-right part is labeled  $\mathbf{u}_{12}$ . The bottom-left part is labeled  $\mathbf{l}_{21}$ . The bottom-right part is labeled  $A_{22}$ .