Matrix Conditioning

$A \in \mathbb{R}^{n \times n}$

$\kappa(A) = \|A\|_2 \|A^{-1}\|_2$

$Ax = b$

Solving for x has condition number $\kappa(A)$

Multiplying x by A has condition number $\kappa(A)$

$$\kappa(A) = \kappa(A^{-1}) = \|A\|_2 \cdot \|A^{-1}\|_2$$

The same problem as solving for b in $A^{-1}b = x$

More Properties of the Condition Number

What is $\mathrm{cond}(A^{-1})$?

$\kappa(A^{-1}) = \|A^{-1}\| \, \|A\| = \kappa(A)$

What is the condition number of applying the matrix-vector multiplication $A\boldsymbol{x} = \boldsymbol{b}$? (I.e. now $\boldsymbol{x}$ is the input and $\boldsymbol{b}$ is the output)

$\kappa(A)$

$$\text{conditioning} = \max_{\text{pert error}} \frac{\text{rel err in output}}{\text{rel err in input}}$$

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|$$

$\|A\|$ = maximum growth in abs err

$\|A^{-1}\|$ = minimum growth in input

Matrices with Great Conditioning (Part 1)

Give an example of a matrix that is *very* well-conditioned.
(I.e. has a condition-number that's *good* for computation.)
What is the best possible condition number of a matrix?

$\kappa(I) = 1$

$\kappa\left(\begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix}\right) = 1$

Orthogonal matrix Q

$Q^T Q = I \Rightarrow Q^T = Q^{-1}$

$\|Q\|_2 = \sqrt{\langle xQ^T, Qx \rangle} = \sqrt{\langle x, x \rangle} = 1$

$\max \|x\|_2 = 1$

$\kappa(Q)$
$= \|Q\|_2 \cdot \|Q^{-1}\|_2$
$= 1 \cdot \|Q^T\|_2$
$= 1 \cdot 1 = 1$

$$\|A \cdot A^{-1}\| = 1$$

$$\|A \cdot A^{-1}\| \leq \|A\| \cdot \|A^{-1}\|$$

$$\|X \cdot Y\| \leq \|X\| \cdot \|Y\|$$

submult

$$\kappa(A) = \|A\| \cdot \|A^{-1}\| \geq 1$$

for any p-norm

Matrices with Great Conditioning (Part 2)

What is the 2-norm condition number of an orthogonal matrix A?

$\|A\|_2 = 1$

In-class activity: Matrix Conditioning

Outline

Solving Systems of Equations

Want methods/algorithms to solve linear systems. Starting small, a kind of system that's easy to solve has a ... matrix.

$$a_{11}x_1 + a_{21}x_2 = b_1$$

$$a_{22}x_2 = b_2$$

given

$$x_2 = \frac{b_2}{a_{22}}$$

$$a_{11}x_1 = b_1 - a_{21} \cdot \left(\frac{b_2}{a_{22}}\right)$$

Triangular Matrices

Solve

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

$$x_4 = b_4 / a_{44}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 - a_{14} \cdot x_4 \\ b_2 - a_{24} \cdot x_4 \\ b_3 - a_{34} \cdot x_4 \end{pmatrix}$$

Demo: Back-substitution

In-class activity: Forward-substitution

$$\begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

General Matrices

What about non-triangular matrices?

Gaussian Elimination = LU

Given $n \times n$ matrix A, obtain lower triangular matrix L and upper triangular matrix U such that $A = LU$.

Is there some redundancy in this representation?

$$n(n+1)/2 = \sum_{i=1}^{n} i = \frac{n}{2} \cdot (n+1)$$

$$1 + 2 + 3 + 4 + 5 + 6 = 3 \cdot 7$$

diagonal of L is 1

"unit diagonal"

Using LU Decomposition to Solve Linear Systems

Given $A = LU$, how do we solve $A\boldsymbol{x} = \boldsymbol{b}$?

$$L \cdot \underbrace{U \cdot x}_{y} = b$$

$L \cdot y = b \Leftarrow$ solve by fwd. subs.

$U \cdot x = y \Leftarrow$ solve for x by bwd. subs.

$$x = U^{-1} L^{-1} b$$

do not compute but apply

2-by-2 LU Factorization (Gaussian Elimination)

Lets consider an example for $n = 2$.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} \\ 0 \end{bmatrix} = \begin{bmatrix} u_{11} \\ l_{21} u_{11} \end{bmatrix}$$

$$a_{21} = l_{21} u_{11} \implies a_{21} \cdot u_{11}^{-1}$$

$$= a_{21} / a_{11}$$

$$a_{22} = \begin{bmatrix} l_{21} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}$$

$$a_{22} = l_{21} \cdot u_{12} + u_{22}$$

$$u_{22} = a_{22} - \underbrace{l_{21} \cdot u_{12}}_{\text{Schur complement}}$$

General LU Factorization (Gaussian Elimination)

$$A = \begin{bmatrix} a_{11} & \mathbf{a_{12}} \\ \mathbf{a_{21}} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{l_{21}} & L_{22} \end{bmatrix} \cdot \begin{bmatrix} u_{11} & \mathbf{u_{12}} \\ 0 & U_{22} \end{bmatrix}$$