Overview

LU

Eigenvalues

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$PA = LU$

$$\begin{pmatrix} 0.00001 & 1 & \cdots \\ 2 & & \\ 3 & & \\ 4 & & \\ 5 & & \\ 1{,}000{,}000 & & \end{pmatrix}$$

In partial piv swap the biggest entry to top

$$\rightarrow \begin{pmatrix} 6 & & \\ 0 & & \\ 0 & & \\ 0 & & \\ 0 & & \end{pmatrix}$$

General LU Partial Pivoting

What does the overall process look like?

(and what pivot do we pick if all values in the column are nonzero) → the largest in abs. val

$$P_1 A = \begin{pmatrix} 1 & 0 \\ l_{21} & L_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & U_{22} \end{pmatrix}$$

Recurse to produce

$$\bar{P} A_{22} = \bar{L}_{22} \bar{U}_{22}$$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \bar{P} \end{pmatrix} P_1$$

$PA = LU$

Why LU?

$Ax = b$

$PA = LU$

$A = P^{-1}LU$

$= P^T LU$

$$\underbrace{P^T L U}_{A} x = b$$

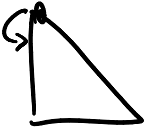
$$LUx = P'b$$

$$x = \underline{U^{-1}L^{-1}\underbrace{P'b}}$$

How do I compute

$L\vec{x} = \vec{y} \iff L^{-1}\vec{y} = \vec{x}$ (w/ $\vec{y} = P'b$)

forw subst



$\tilde{x} = \tilde{y}$

$$\begin{pmatrix} 0 & 0 & 0 & & 1 \\ & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & 1 & \end{pmatrix} \begin{pmatrix} \tilde{x} \end{pmatrix}$$

Apply P in just O(n)

Given an LU factorization

$PA = LU$, can solve

$Ax = b$ in $O(n^2)$

More cost concerns

What's the cost of solving $Ax = b$? (given LU)

$O(n^2)$

What's the cost of solving $Ax_1 = b_1, \ldots, Ax_n = b_n$? (given LU)

$O(n^3) = O(n^2) \cdot O(n)$

What's the cost of finding A^{-1}?

RHS

What's the cost of n solves? (not given LU)

$O(n^3) + O(n^3) = O(n^3)$

$$A x_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$A x_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} & X & \\ | & | & | \\ x_1 & x_n & \end{pmatrix} = I \rightsquigarrow X = A^{-1}$$

$\hookrightarrow O(n^3)$

$$\det(A) = \det(P^T L U)$$

$$= \underset{\pm 1}{\det(P)} \cdot \underset{1}{\det(L)} \cdot \underset{\prod u_{ii}}{\det(U)}$$

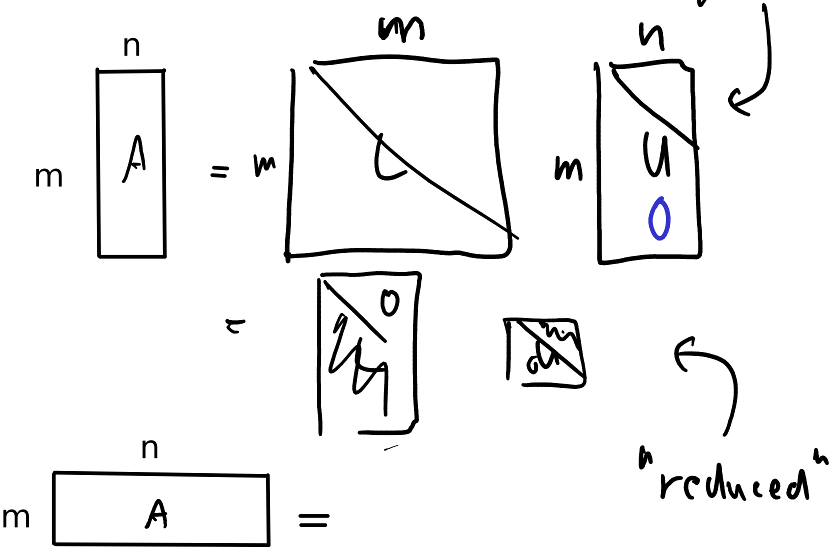
$$\det(AB) = \det(A)\det(B)$$

$$\begin{array}{cc} & n \\ m & A \end{array} = L \qquad U$$

$$m \times ? \qquad ? \times n$$

LU: Rectangular Matrices

> Can we compute LU of an $m \times n$ rectangular matrix? Full



Outline

- Python, Numpy, and Matplotlib
- Making Models with Polynomials
- Making Models with Monte Carlo
- Error, Accuracy and Convergence
- Floating Point
- Modeling the World with Arrays
  - The World in a Vector
  - What can Matrices Do?
  - Graphs
  - Sparsity
- Norms and Errors
- The 'Undo' Button for Linear Operations: LU
- Repeating Linear Operations: Eigenvalues and Steady States
- Eigenvalues: Applications
- Approximate Undo: SVD and Least Squares
- SVD: Applications
  - Solving Funny-Shaped Linear Systems
  - Data Fitting
  - Norms and Condition Numbers
  - Low-Rank Approximation
- Interpolation
- Iteration and Convergence
- Solving One Equation
- Solving Many Equations
- Finding the Best: Optimization in 1D
- Optimization in n Dimensions

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- $\boldsymbol{x} \neq \boldsymbol{0}$ is called an eigenvector of A if there exists a λ so that

$$A\boldsymbol{x} = \lambda \boldsymbol{x}.$$

- In that case, λ is called an eigenvalue.
- By this definition if $\boldsymbol{x}$ is an eigenvector then so is $\alpha \boldsymbol{x}$, therefore we will usually seek normalized eigenvectors, so $\|\boldsymbol{x}\|_2 = 1$.

$$17\vec{x} = \vec{y}$$

$$Ay = 17A\vec{x} = 17\lambda\vec{x} = \lambda\vec{y}$$

Finding Eigenvalues

How do you find eigenvalues?

$\det(A - \lambda I)$

$=$ characteristic polynomial$(\lambda$

$\to$ solve $CP(\lambda) = 0$

for eigenvalues λ.

Polynomials of degree 5 or higher may not have a closed-form solution.

→ comp. consequence:
no algorithm with finite # of steps.

Distinguishing eigenvectors

$A\vec{x}_1 = \lambda_1 \vec{x}_1$

> Assume we have normalized eigenvectors $x_1, \ldots, x_n$ with eigenvalues $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Show that the eigenvectors are linearly-independent.

$\vec{x}$

$$\vec{0} = \alpha_1 \vec{x}_1 + \alpha_2 x_2 + \cdots + \alpha_n \vec{x}_n$$

To show: $\alpha_1 = \cdots = \alpha_n = 0$.

$$A\vec{x} = \alpha_1 A\vec{x}_1 + \cdots + \alpha_n A\vec{x}_n$$

$$= \alpha_1 \lambda_1 \vec{x}_1 + \alpha_2 \lambda_2 x_2 + \cdots + \alpha_n \lambda_n \vec{x}_n$$

$\alpha_2 \lambda_2$

$$\frac{A\vec{x}}{\lambda_1} = \underline{\alpha_1 \vec{x}} + \alpha_2 \underbrace{\frac{\lambda_2}{\lambda_1}}_{<1} \vec{x}_2 + \cdots + \frac{\lambda_n}{\lambda_1} \vec{x}_n$$

$$\frac{A^{\overbrace{10{,}000}^{l}} \vec{x}}{\lambda_1^{10{,}000}} = \vec{\alpha}_1 \vec{x} + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{10{,}000} \vec{x}_2 + \cdots$$

$$\lim_{l \to \infty} 0 = \frac{A^l \vec{x}}{\lambda_1^l} = \alpha_1 \vec{x}_1$$

$\Rightarrow \alpha_1 = 0$

Ind. $\alpha_2 = \cdots \alpha_n = 0$

Diagonalizability

If we have n eigenvectors with different eigenvalues, the matrix is diagonalizable.

Are all Matrices Diagonalizable?

Give characteristic polynomial, eigenvalues, eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}.$$

Power Iteration

We can use linear-independence to find the eigenvector with the largest eigenvalue. Consider the eigenvalues of A^{1000}.

Power Iteration: Issues?

What could go wrong with Power Iteration?

What about Eigenvalues?

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

$$Ax = \lambda x$$

$$\frac{x^T A x}{x^T x} = \frac{\lambda x^T x}{x^T x} = \lambda$$

Convergence of Power Iteration

What can you say about the convergence of the power method?
Say $\boldsymbol{v}_1^{(k)}$ is the kth estimate of the eigenvector $\boldsymbol{x}_1$, and

$$e_k = \left\| \boldsymbol{x}_1 - \boldsymbol{v}_1^{(k)} \right\|.$$