Reconstructing a Function From Derivatives

Found: *Taylor series approximation*.

$$f(0+x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots$$

The general Taylor expansion with center $x_0 = 0$ is

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

Demo: Polynomial Approximation with Derivatives (Part I)

Shifting the Expansion Center

- Can you do this at points other than the origin?

Errors in Taylor Approximation (I)

- Can't sum infinitely many terms. Have to *truncate*. How big of an error does this cause?

Demo: Polynomial Approximation with Derivatives (Part II)

$$T(n,x) = a + bx \dots + cx^n$$

$$T(n+1,x) = a \dots \quad dx^{n+1}$$

$$E_n(x) = f(x) - T(n,x) \leq C \cdot x^{n+1}$$

$$= O(x^{n+1})$$

$$x = x_0 + h \qquad \text{if } x < 1$$

$$E_n(h) = f(x_0+h) - T(n, x_0+h) = O(h^{n+1})$$

Making Predictions with Taylor Truncation Error

- Suppose you expand $\sqrt{x-10}$ in a Taylor polynomial of degree 3 about the center $x_0 = 12$. For $h_1 = 0.05$, you find that the Taylor truncation error is about 10^{-4}.

  What is the Taylor truncation error for $h_2 = 0.025$?

$$E(h) \approx C h^{n+1} \approx C h^4$$

$$E(h_2) = C h_2^4 = C \left(\frac{h_2^4}{h_1^4}\right) h_1^4 = E(h_1) \left(\frac{h_2}{h_1}\right)^4$$

$$E(h_1) = E(2h_2)$$

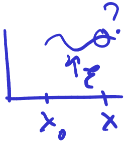
$$= \left(\frac{1}{2}\right)^4 E(h_1)$$

Demo: Polynomial Approximation with Derivatives (Part III)

Taylor Remainders: the Full Truth

Let $f : \mathbb{R} \to \mathbb{R}$ be $(n+1)$-times differentiable on the interval (x_0, x) with $f^{(n)}$ continuous on $[x_0, x]$. Then there exists a $\xi \in (x_0, x)$ so that

$$f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}}_{\text{“}C\text{”}} \cdot (\xi - x_0)^{n+1}$$



and since $|\xi - x_0| \leqslant h$

$$\left| f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i \right| \leqslant \underbrace{\frac{\left|f^{(n+1)}(\xi)\right|}{(n+1)!}}_{\text{“}C\text{”}} \cdot h^{n+1}.$$

$O(h^{n+1})$

Proof of Taylor Remainder Theorem

◦ Intuitively the error of an approximation that takes into account n derivatives should be proportional to the maximum value of the $(n+1)$th one...

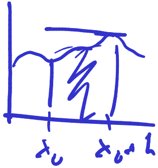
$$E(h) = |f(x_0+h) - T_n(x_0+h)|$$

$$E^{(n+1)}(h) = |f^{(n+1)}(x_0+h)|$$

$$E(h) = \int_{x_0}^{x_0+h} \cdots \int E^{(n+1)}(h)\, dx^{n+1}$$

$$= \int_{x_0}^{x_0+h} \cdots \int |f^{(n+1)}(x_0+h)|\, dx^{n+1}$$

$$\le \int_{x_0}^{x_0+h} \cdots \int \max_{\xi \in [x_0, x_0+h]} |f^{(n+1)}(\xi)|\, dx^{n+1}$$



In-class activity: Taylor series

Using Polynomial Approximation

- Suppose we can approximate a function as a polynomial:

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

How is that useful? Say, if I wanted the integral of f?

$$\int_s^t f(x)\,dx = \int_s^t a_0 + \int_s^t a_1 x + \dots$$

Demo: Computing π with Taylor