Outline

- Review
  - Power Iteration
  - Inverse Iteration
  - Rayleigh Quotient Iteration
- Convergence
- How to get all the eigenvectors
- Applications of E-V problems

Exam 1 ed

Matrix Norms / conditioning

LU factorization

pivoting

Power Iteration / Diagonalization

$$
\begin{array}{cc}
 & \begin{array}{cc} n/2 & n/2 \end{array} \\
\begin{array}{c} n/2 \\ n/2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\end{array}
$$

$$
C = AB \rightarrow \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}
$$

$$
C_{11} = \underbrace{A_{11} B_{11}}_{\text{compute}} + A_{12} B_{21}
$$

...

eig vecs $\underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{X}$ $\underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{D}$

$$AX = \begin{bmatrix} | & & | \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ | & & | \end{bmatrix}$$

$$= XD$$

$$\Rightarrow X^{-1}AX = D \quad \Rightarrow \underbrace{A = XDX^{-1}}$$

$$x = \frac{A^k x_0}{\|A^k x_0\|}$$

$k \to \infty$

$$\underline{x_0} = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3$$

λ_1 λ_2 λ_3

-5 1 5

Convergence of Power Iteration

What can you say about the convergence of the power method? Say $\boldsymbol{v}_1^{(k)}$ is the kth estimate of the eigenvector $\boldsymbol{x}_1$, and

$$e_k = \left\| \boldsymbol{x}_1 - \boldsymbol{v}_1^{(k)} \right\|.$$

$$v_1^{(0)} = \alpha_1 x_1 + \alpha_2 x_2$$

$$\lambda_1 > \lambda_2$$

Convergence rate

$$v_1^{(k+1)} = \lambda_1^k \left(\alpha_1 x_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 \right)$$

$$\ll 1$$

Inverse Iteration / Rayleigh Quotient Iteration

Describe inverse iteration.

$$v^{(k+1)} = A^{-1} v^{(k)}$$

Describe Rayleigh Quotient Iteration. σ

$$v^{(k+1)} = (A - I\sigma)^{-1} v^{(k)}$$

1. Compute LU
2. Solve tri. linear system k-times

Computing Multiple Eigenvalues

All Power Iteration Methods compute one eigenvalue at a time.
What if I want *all* eigenvalues?

Simultaneous Iteration

1. $X^{(k+1)} = A X^{(k)}$

   where $X^{(0)}$ is random

2. orthogonalize the columns of X

   - that columns of X are orthogonal

     Gram-Schmidt

$$\begin{array}{cc} & X \\ \left[\quad\quad\right] & \left[\ \cdots\ \right] \end{array}$$

A Iterative Variant

$$x_1 = \frac{A^{100}\bar{x}_1}{\|A^{100}\bar{x}\|} \qquad x_1 - \text{largest e.g. vec.}$$

pick $\bar{x}_2$ randomly with $\|\bar{x}_2\|_2 = 1$ and orthogonalize $\bar{x}_2, \bar{x}_1$

$$x_{2\text{-orth}} = \bar{x}_2 - \underbrace{\langle \bar{x}_2, x_1 \rangle}_{x_2^T x_1} x_1$$

Have largest eigenvectors $\{x_1 \ldots x_k\}$
Want to find x_{k+1}

1. Pick random $\bar{x}$ that is orthogonal to the span($\{x_1 \ldots x_k\}$).
via Gram-Schmidt
2. Run power iteration with $\bar{x}$ to get x_{k+1} (reorthogonalize)

1. Deflation

Find Q orthogonal

$$\underline{Q^T A Q} \Rightarrow \begin{bmatrix} \lambda_1 & \sim\sim\sim \\ \vdots & \sim\sim\sim \\ 0 & \sim\sim\sim \\ \vdots & \sim\sim\sim \end{bmatrix}$$

$$\underline{Q^T} \begin{pmatrix} 1 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}$$

Symmetric matrices

$A = QDQ^T$ ← always

$\bar{A} = A$

For $i = 1$ to n

find orthogonal matrix Q_i

Householder

$Q_i = \ldots$

s.t. $Q_i^T \bar{A} Q_i = \begin{bmatrix} \lambda & 0 \\ 0 & \boxed{} \end{bmatrix}$

continue with $\bar{A} = Q_i^T \bar{A} Q_i$

$$A = \underbrace{Q_n^T \cdots Q_1^T}_{Q^T} A \underbrace{Q_1 \cdots Q_n}_{Q}$$