

Outline

Review

Power Iteration

Inverse Iteration

Rayleigh Quotient Iteration

Convergence

How to get all the eigenvectors

Applications of E-V problems

Exam 1st

Matrix Norms / Cond. number

LU factorization

pivoting

Power Iteration / Diagonalization

$$\begin{matrix} & n/2 & n/2 \\ n/2 & \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right] \end{matrix}$$

$$C = AB \rightarrow \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$C_{11} = A_{11} B_{11} + A_{12} B_{21}$$

...

↓
compute

eig
vecs

$$\underbrace{\begin{bmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{bmatrix}}_X$$

$$\underbrace{\begin{bmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_n \end{bmatrix}}_D$$

$$AX = \begin{bmatrix} | & & | \\ \lambda_1 x_1 & \dots & \lambda_n x_n \\ | & & | \end{bmatrix}$$

$$= XD$$

$$\Rightarrow X^{-1}AX = D \quad \Rightarrow A = \underbrace{XD X^{-1}}$$

$$x = \frac{A^k x_0}{\|A^k\|} \quad \Bigg| \quad k \rightarrow \infty$$

$$\underline{x_0} = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3$$

Diagram illustrating the decomposition of x_0 into eigenvectors w_1, w_2, w_3 with corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$ shown above the terms. Arrows point from the eigenvalues to their respective eigenvectors.

Convergence of Power Iteration

What can you say about the convergence of the power method?

Say $\mathbf{v}_1^{(k)}$ is the k th estimate of the eigenvector \mathbf{x}_1 , and

$$e_k = \left\| \mathbf{x}_1 - \mathbf{v}_1^{(k)} \right\|.$$

$$\mathbf{v}_1^{(0)} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2$$

$$|\lambda_1| > |\lambda_2|$$

$$\mathbf{v}_1^{(k+1)} = \frac{1}{\lambda_1^k} \left(\alpha_1 \mathbf{x}_1 + \alpha_2 \left(\frac{|\lambda_2|}{|\lambda_1|} \right)^k \mathbf{x}_2 \right)$$

Converge
v2/v1
↓
≪ 1

Inverse Iteration / Rayleigh Quotient Iteration

Describe inverse iteration.

$$v^{(k+1)} = A^{-1} v^{(k)}$$

Describe Rayleigh Quotient Iteration.

$$v^{(k+1)} = (A - \sigma I)^{-1} v^{(k)}$$

1. Compute σ
2. Solve k linear systems k -times

Computing Multiple Eigenvalues

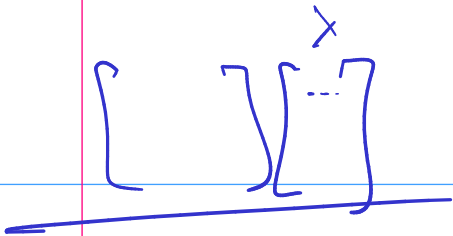
All Power Iteration Methods compute one eigenvalue at a time.
What if I want *all* eigenvalues?

Simultaneous Iteration

$$1. X^{(k+1)} = A X^{(k)}$$

where $X^{(0)}$ is random

2. or the given A , the columns of X
- that columns of X are orthogonal
Gram-Schmidt



A 1 kernel Variant

$$x_1 = \frac{A^{100} \bar{x}_1}{\|A^{100} \bar{x}_1\|} \quad x_1 - \text{largest e.g. rec.}$$

p. 46 \bar{x}_2 randomly with $\|\bar{x}_2\|_2 = 1$ and orthogonalize \bar{x}_2, \bar{x}_1

$$x_{2\text{-orth}} = \bar{x}_2 - \frac{\langle \bar{x}_2, x_1 \rangle}{x_2^T x_1} x_1$$

Have largest eigenvalues $\{x_1, \dots, x_k\}$

Want to find x_{k+1}

1. Pick random \bar{x} that is orthogonal to the span $(\{x_1, \dots, x_k\})$.
via Gram-Schmidt
2. Run power iteration with \bar{x} to get x_{k+1} (rec + orthogonalize)

1. Deflation

Find Q column!

$$\underbrace{Q^T A Q}_{\substack{\uparrow \\ \text{1st} \\ \text{col}}} \Rightarrow \begin{bmatrix} \lambda_1 & & \\ & \sim & \\ & & \sim \end{bmatrix}$$

Symmetric matrices

$$A = Q D Q^T \leftarrow \text{always}$$

$$\bar{A} = A$$

For $i = 1$ to n

Householder find orthogonal matrix Q_i

$$Q_i = \begin{bmatrix} | & & \\ - & & \\ | & & \end{bmatrix}$$

s.t

$$Q_i^T \bar{A} Q_i = \begin{bmatrix} \lambda & & & \\ & \square & & \\ & & \ddots & \\ & & & \lambda \end{bmatrix}$$

combine with $\bar{A} = Q_i^T \bar{A} Q_i$

$$A = \underbrace{Q_n^T \dots Q_1^T}_{Q^T} A \underbrace{Q_1 \dots Q_n}_Q$$