Interpolation

Accuracy

Monomial Basis Problems

Orthogonal Bases

Chebyshev

choice of nodes

choice of fun's

Outline

Recap: Interpolation

Starting point: Looking for a linear combination of functions φ_i to hit given data points (x_i, y_i).

Interpolation becomes solving the linear system:

$$y_i = f(\underset{\text{nodes}}{x_i}) = \sum_{j=0}^{N_{\text{func}}} \underset{\text{coeffs}}{\alpha_j} \underbrace{\varphi_j(x_i)}_{V_{ij}} \quad \leftrightarrow \quad V\boldsymbol{\alpha} = \boldsymbol{y}.$$

Want unique answer: Pick $N_{\text{func}} = N \to V$ square.

V is called the (generalized) Vandermonde matrix.

Main lesson:

$$V\,(\text{coefficients}) = (\text{values at nodes})\,.$$

Rethinking Interpolation

We have so far always used monomials $(1, x, x^2, x^3, \ldots)$ and equispaced points for interpolation. It turns out that this has *significant problems*.

Demo: Monomial interpolation

Demo: Choice of Nodes for Polynomial Interpolation

Interpolation: Choosing Basis Function and Nodes

Both function basis and point set are under our control. What do we pick?

Ideas for basis functions:

- Monomials $1, x, x^2, x^3, x^4, \ldots$
- Functions that make $V = I \to$ 'Lagrange basis'
- Functions that make V triangular $\to$ 'Newton basis'
- Splines (piecewise polynomials)
- Orthogonal polynomials
- Sines and cosines
- 'Bumps' ('Radial Basis Functions')

$(x-x_1)(x-x_2)$
$(x-x_2)(x-x_3)$
$(x-x_1)(x-x_3)$

Ideas for nodes:

- Equispaced
- 'Edge-Clustered' (so-called Chebyshev/Gauss/... nodes)

Better Conditioning: Orthogonal Polynomials $x^{50} \approx \frac{1}{2}x^{49} + \frac{1}{2}x^{51}$

What caused monomials to have a terribly conditioned Vandermonde?

functions are almost linearly dependent

What's a way to make sure two vectors are *not* like that?

orthogonality

But polynomials are functions!

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

$$\langle f, g \rangle_w = \int w(x)\, f(x)\, g(x)\, dx$$

Legendre Polynomials

start: $[1, x, x^2, x^3]$

orthogonalize f w.r.t. g

$$\bar{f}(x) = f(x) - \langle f, g \rangle \cdot g(x)$$

orthogonalize monomial w.r.t. previous

Better Conditioning: Orthogonal Polynomials (II)

But how can I practically compute the Legendre polynomials?

Look them up!

3-term recurrence

$L_i = f(L_{i-1}, L_{i-2}, L_{i-3})$

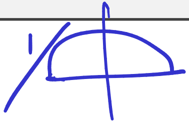
Another Family of Orthogonal Polynomials: Chebyshev

Three equivalent definitions: $w(x)$ $\langle f, g \rangle_w = \int w \cdot g \cdot f$

- Result of Gram-Schmidt with weight $1/\sqrt{1-x^2} = y$

What is that weight?

$y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$



- $T_k(x) = \underline{\cos(k \cos^{-1}(x))}$
- $T_k(x) = \underline{2xT_{k-1}(x) - T_{k-2}(x)}$

Demo: Chebyshev interpolation part I

What are good nodes to use with Chebyshev polynomials?

$x_i = \cos\left(\frac{i}{n}\pi\right)$

$T_k(x_i) = \cos\left(\frac{ik}{n}\pi\right)$ ← equispaced

Chebyshev Nodes

Might also consider zeros (instead of roots) of T_k:

$$x_i = \cos\left(\frac{2i+1}{2k}\pi\right) \quad (i = 1\ldots,k).$$

The Vandermonde for these (with T_k) can be applied in $O(N \log N)$ time, too.

It turns out that we were still looking for a good set of interpolation nodes.

We came up with the criterion that the nodes should bunch towards the ends. Do these do that?

Demo: Chebyshev interpolation part II

$$Vz = y \qquad \begin{bmatrix} f_0(x_0) & \cdots & f_{n-1}(x_0) \\ \vdots & & \\ f_0(x_{n-1}) & & \end{bmatrix}$$

$\uparrow$ solve

given $\bar{x}$ (some point)

$$\tilde{f}(\bar{x}) = \sum_i z_i \cdot f_i(\bar{x})$$

$$= z^T \cdot \begin{bmatrix} f_0(\bar{x}) \\ \vdots \\ f_{n-1}(\bar{x}) \end{bmatrix}$$

Calculus on Interpolants

Suppose we have an interpolant $\tilde{f}(x)$ with $f(x_i) = \tilde{f}(x_i)$ for $i = 1, \ldots, n$:

$$\tilde{f}(x) = \alpha_1 \varphi_1(x) + \cdots + \alpha_n \varphi_n(x)$$

How do we compute the derivative of $\tilde{f}$?

$$\tilde{f}'(x) = \alpha_1 \varphi_1'(x) + \cdots + \alpha_n \varphi_n'(x)$$

$$f'(x) \approx \tilde{f}'(x)$$

Suppose we have function values at nodes $(x_i, f(x_i))$ for $i = 1, \ldots, n$ for a function f. If we want $f'(x_i)$, what can we do?

exact is hard

$$z = V(x)^{-1} f(x) \Rightarrow \tilde{f}(x)$$

differentiate $\tilde{f}(x)$