

# Quadrature

given weights from  $[0, 1]$   
evaluate integral on  $[a, b]$

Error (Simpson's rule)

## Convergence Rates

Power Iteration vs Rayleigh Quotient  
times                      quadratic

Nonlinear Solve

## Using Quadrature Rules

To estimate an integral over an arbitrary interval  $[a, b]$  we can use a quadrature rule with weights derived by integrating over  $[0, 1]$ , since

$$\int_a^b f(x) dx = (b-a) \int_0^1 f((b-a)\bar{x} + a) d\bar{x}.$$

Thus, given weights  $\mathbf{w} = V^{-T} \mathbf{d}$  computed from integrating  $n$  basis functions on  $[0, 1]$  (to get  $\mathbf{d}$ ) and  $V$  defined based on points  $\bar{x}_1, \dots, \bar{x}_n \in [0, 1]$ , we can use the same weights for the above integral as

$$\int_a^b f(x) dx \approx (b-a) \mathbf{w}^T \mathbf{y}.$$

Above  $\mathbf{y}$  corresponds to  $f$  evaluated at points  $(b-a)\bar{x}_1 + a, \dots, (b-a)\bar{x}_n + a$ .

Vandermonde matrix  $V$

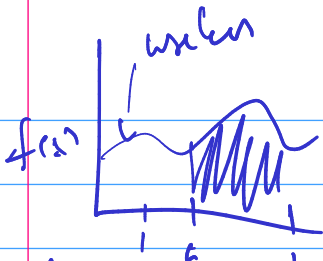
based on points  $x_1, \dots, x_n \in [0, 1]$

$$\int_0^1 \varphi_i(x) dx = d_i$$

$$\int_0^1 f(x) dx = a_0 + a_1 d_1 + \dots + a_n d_n$$

if we want  $\downarrow$  we need

$f(x_1) \dots f(x_n)$



$d_i \rightarrow$  basis function integrals

$w_i \rightarrow$  weights can be computed

based on  $[0, 1]$

$b$

$a$

$$\int_a^b f(x) dx \approx \sum w_i^T y_i$$

$(a, b)$

$y_i$

$d_i$ : integrals of basis fns

$x_i$ : nodes define Vandermonde  $V$

## Facts about Quadrature

$$w = \left[ \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right]$$

What does Simpson's rule look like on  $[0, 1/2]$ ?

$$b = \frac{1}{2}$$
$$a = 0$$

$$\left( \frac{1}{2} w \right)$$

What does Simpson's rule look like on  $[5, 6]$ ?

weights still  $w$

$$y = \begin{bmatrix} f(5) \\ f(5.5) \\ f(6) \end{bmatrix}$$

$$int = w^T y$$

How accurate is Simpson's rule with polynomials of degree  $n$ ?

Accuracy of Simpson's rule with  
polynomial basis of degree  $n$

Interpolation error on interval  
 $[a, b]$  where  $b - a = h$

$$\tilde{f}(x) - f(x) = O(h^{n+1})$$

$\uparrow$   
in  $[a, b]$

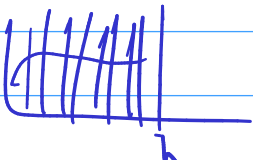
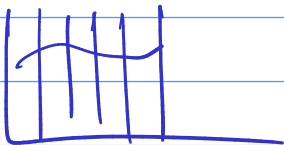
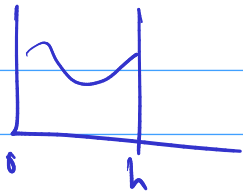
Simpson's rule (integration)

$$\left| \int_a^b f(x) dx - \int_c^b \tilde{f}(x) dx \right| = O(h^{n+2})$$

as  $h$  decreases  $O(h^{n+2})$

decreases faster than  $O(h^{n+1})$

$$\left| f'(x) - \tilde{f}'(x) \right| = O(h^n)$$





# Outline

Python, Numpy, and Matplotlib  
Making Models with Polynomials  
Making Models with Monte Carlo  
Error, Accuracy and Convergence  
Floating Point  
Modeling the World with Arrays  
    The World in a Vector  
    What can Matrices Do?  
    Graphs  
    Sparsity  
Norms and Errors  
The 'Undo' Button for Linear Operations: LU  
Repeating Linear Operations:  
Eigenvalues and Steady States  
Eigenvalues: Applications

Approximate Undo: SVD and Least Squares

SVD: Applications

Solving Funny-Shaped Linear Systems

Data Fitting

Norms and Condition Numbers

Low-Rank Approximation

Interpolation

Iteration and Convergence

~~Solving One Equation~~

~~Solving Many Equations~~

Finding the Best: Optimization in 1D

Optimization in  $n$  Dimensions

What is linear convergence? quadratic convergence? iterative methods

linear convergence: decrease error by a constant at every iteration

error at  $k^{\text{th}}$  iteration

$$e_k \approx \frac{1}{c} e_{k-1}$$

$$\lim_{k \rightarrow \infty} e_k / e_1^k = C$$

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Quadratic convergence: square the error from the previous,  $e_k \approx \frac{1}{c} e_{k-1}^2$

E: Most important advice  
to future students in CS 357.

F: Rank in interest (high is better)

(1.) Taylor Expansion

(2.) Interpolation

(3.) Monte Carlo

(4.) SVD