Quadrature

given weights from [0, 1]

evaluate integral or [a, b]

Error (Simpson's rule)

Convergence Rates

$\underbrace{\text{Power Iteration}}_{\text{linear}}$ vs $\underbrace{\text{Rayleigh Quotient}}_{\text{quadratic}}$

Nonlinear Solve

Using Quadrature Rules

To estimate an integral over an arbitrary interval $[a, b]$ we can use a quadrature rule with weights derived by integrating over $[0, 1]$, since

$$\int_a^b f(x)dx \underbrace{=}_{x=(b-a)\bar{x}+a} (b-a)\int_0^1 f((b-a)\bar{x}+a)d\bar{x}.$$

Thus, given weights $\mathbf{w} = V^{-T}\mathbf{d}$ computed from integrating n basis functions on $[0, 1]$ (to get $\mathbf{d}$) and V defined based on points $\bar{x}_1, \ldots, \bar{x}_n \in [0, 1]$, we can use the same weights for the above integral as

$$\int_a^b f(x)dx \approx (b-a)\mathbf{w}^T\mathbf{y}.$$

$V^{-T} d = w$

$(x_i, y_i) \in 0,1$

Above $\mathbf{y}$ corresponds to f evaluated at points $(b-a)\bar{x}_1 + a, \ldots, (b-a)\bar{x}_n + a$.

y V d

integral,

Vandermonde matrix V

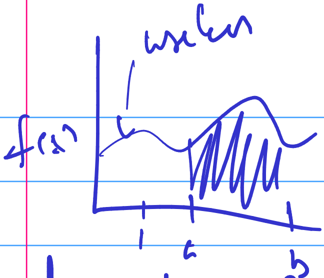
based on points $x_1 \dots x_n \in [0,1]$

$\int_0^1 \varphi_i(x)\, dx = d_i$

$\int_0^1 f(x)\, dx = a_1 d_1 + \dots + a_n d_n$

if we want we need

$f(x_1) \dots f(x_n)$



$d_i \to$ basis function integrals

$w_i \to$ weights can be computed based on $[0,1]$

$$\int_a^b f(x)\,dx \approx \underbrace{d\,V}_{w^T} \overset{[a,b]}{\downarrow} y$$

← d_i: integrals of basis fns

← x_i nodes define Vandermonde V

Facts about Quadrature

$w = \left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right]$

What does Simpson's rule look like on $[0, 1/2]$?

$b = \frac{1}{2}$

$a = 0$

$\frac{1}{2} w$

What does Simpson's rule look like on $[5, 6]$?

weights still w

$y = \begin{bmatrix} f(5) \\ f(5.5) \\ f(6) \end{bmatrix}$

$\text{int} = w^T y$

How accurate is Simpson's rule with polynomials of degree n?

Accuracy of Simpson's rule with polynomial basis of degree n

Interpolation error on interval $[a,b]$ where $b-a=h$

$$\tilde{f}(x)-f(x)=O(h^{n+1})$$

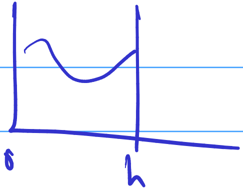
$\uparrow$ in $[a,b]$

Simpson's rule (integration)

$$\left| \int_a^b f(x)\,dx - \int_a^b \tilde{f}(x)\,dx \right| = O(h^{n+2})$$

as h decreases $O(h^{n+2})$
decreases faster than $O(h^{n+1})$

$$\left| f'(x) - \tilde{f}'(x) \right| = O(h^n)$$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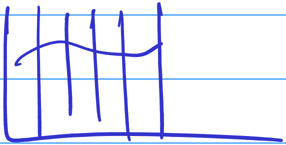

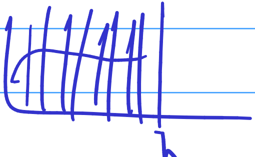
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Outline

What is linear convergence? quadratic convergence? iterative methods

linear convergence: decrease error by a constant at every iteration

error at k^{th} iteration

$$e_k \approx \frac{1}{c} e_{k-1}$$

$$\lim_{k \to \infty} e_k / e_1^k = c$$

Quadratic convergence: square the error from the previous, $e_k \approx \frac{1}{c} e_{k-1}^2$

E: Most important advice to future students in CS 357.

F: Rank in interest (high is better)

(1.) Taylor Expansion
(2.) Interpolation
(3.) Monte Carlo
(4.) SVD