Overview

- E i yor: app
- SVN
- C sq
Recall our example of a Markov chain:

Suppose this is an accurate model of the behavior of the average student. :) How likely are we to find the average student in each of these states?
Demo: Finding an equilibrium distribution using the power method
Many important systems in nature are modeled by describing the time rate of change of something.

E.g. every bird will have 0.2 baby birds on average per year.
But there are also foxes that eat birds. Every fox present decreases the bird population by 1 birds a year.
Meanwhile, each fox has 0.3 fox babies a year. And for each bird present, the population of foxes grows by 0.9 foxes.
Set this up as equations and see if eigenvalues can help us understand how these populations will evolve over time.

\[
\frac{\partial b}{\partial t} + \frac{\partial f}{\partial t} = 0.2b - 1f
\]

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial t} = 0.9b + 0.3f
\]
\[ \dot{S} = \left( \begin{array}{c} b \\ \dot{E} \end{array} \right) \]

\[ \frac{\partial S}{\partial t} = \left( \begin{array}{cc} 0.2 & -1 \\ 0.9 & 0.3 \end{array} \right) S \]

\[ S(t) = e^{At} \cdot S_0 \]

\[ \lambda e^{At} S_0 = A e^{At} S_0 \]
Demo: Understanding the birds and the foxes with eigenvalues

In-class activity: Eigenvalues 2
\[ e^{it} = \cos(it) + is\sin(it) \]
Outline

- Python, Numpy, and Matplotlib
- Making Models with Polynomials
- Making Models with Monte Carlo
- Error, Accuracy and Convergence
- Floating Point
- Modeling the World with Arrays
  - The World in a Vector
  - What can Matrices Do?
  - Graphs
  - Sparsity
- Norms and Errors
- The ‘Undo’ Button for Linear Operations: LU
- Repeating Linear Operations:
- Eigenvalues and Steady States
- Eigenvalues: Applications
- Approximate Undo: SVD and Least Squares
  - SVD: Applications
    - Solving Funny-Shaped Linear Systems
    - Data Fitting
    - Norms and Condition Numbers
    - Low-Rank Approximation
- Interpolation
- Iteration and Convergence
- Solving One Equation
- Solving Many Equations
- Finding the Best: Optimization in 1D
- Optimization in $n$ Dimensions
Singular Value Decomposition

What is the Singular Value Decomposition (‘SVD’)?

\[ A = U \Sigma V^T \]

- **Full**: \( A^{\text{full}} : m \times n \)
- **U**: \( U^{\text{full}} : m \times m \)
- **\Sigma**: \( \Sigma^{\text{full}} : m \times n \)
- **V^T**: \( V^{\text{full}} : n \times n \)
\[ A x = b \]
\[ A = UV^r \]
\[ U \Sigma V^t x = b \]
\[ \Sigma V^t x = U^t b \]
\[ V^t x = U^t b \]
\[ x = V (U^t b) \]
How can I compute an SVD of a matrix $A$?

$A^T A$ is symmetric, pos. semi-def. $\Rightarrow$ eig. $\geq 0$

$A^T A \mathbf{v}_i = \lambda_i \mathbf{v}_i$

$V = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$

$A^T A V = V \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$
\[ \sigma_i = \sqrt{\lambda_i} \quad \text{"singular value"} \]

\[
\mathbf{e} = \begin{pmatrix}
\sqrt{\lambda_1} \\
\vdots \\
\sqrt{\lambda_n}
\end{pmatrix} = 
\begin{pmatrix}
\sigma_1 \\
\vdots \\
\sigma_n
\end{pmatrix}
\]

\[
A^t A V = V \mathbf{e}^2 \quad \Rightarrow \quad V^t A^t A V = \mathbf{e}^2
\]

"Eigenvectors of a symm. matrix are orth.\ V orth. "
\[ A = U \Sigma V^T \]

\[ \Sigma V C^{-1} = \eta \]

\[ t = U^T u = \underbrace{C^{-1} V^T A^+ A v C^{-1}}_{\mathcal{C}^2} = \Gamma \]
Demo: Computing the SVD
How Expensive is it to Compute the SVD?

**Demo:** Relative Cost of Matrix Factorizations
‘Reduced’ SVD
Is there a ‘reduced’ factorization for non-square matrices?
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Solve Square Linear Systems

Can the SVD $A = UΣV^T$ be used to solve *square* linear systems? At what cost (once the SVD is known)?