Overview:

- Taylor (demo/interactive) (derivative \rightarrow function)
- applying polynomial approximations (compute π)
- interpolation (point values \rightarrow function)
Reconstructing a Function From Derivatives

Found: *Taylor series approximation.*

\[ f(0 + x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots \]

The general Taylor expansion with center \( x_0 = 0 \) is

\[ f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!}x^i \]

**Demo:** Polynomial Approximation with Derivatives (Part I)
Shifting the Expansion Center

○ Can you do this at points other than the origin?

\[ f(x_0 + (x - x_0)) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i \]

\[ h = x - x_0 \]

\[ f(x_0 + h) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!} h^i \]
Errors in Taylor Approximation (I)

- Can’t sum infinitely many terms. Have to *truncating*. How big of an error does this cause?

**Demo:** Polynomial Approximation with Derivatives (Part II)

\[
\left| f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i \right| \leq C \cdot h^{n+1}
\]

\[
= O(h^{n+1})
\]
Suppose you expand $\sqrt{x} - 10$ in a Taylor polynomial of degree 3 about the center $x_0 = 12$. For $h_1 = 0.05$, you find that the Taylor truncation error is about $10^{-4}$.

What is the Taylor truncation error for $h_2 = 0.025$?

\[
\text{Error}(h) \approx C \cdot h^{n+1}
\]

\[
\text{Error}(h_1 = 0.05) \approx 10^{-4} = C \cdot h_1^4
\]

\[
\text{Error}(h_2 = 0.025) \approx C \cdot h_2^4 = C \cdot \left(\frac{h_1}{2}\right)^4
\]

Demo: Polynomial Approximation with Derivatives (Part III)
Taylor Remainders: the Full Truth

Let $f: \mathbb{R} \to \mathbb{R}$ be $(n + 1)$-times differentiable on the interval $(x_0, x)$ with $f^{(n)}$ continuous on $[x_0, x]$. Then there exists a $\xi \in (x_0, x)$ so that

$$f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i = \frac{f^{(n+1)}(\xi)}{(n + 1)!} \cdot (\xi - x_0)^{n+1}$$

and since $|\xi - x_0| \leq h$

$$\left| f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i \right| \leq \frac{|f^{(n+1)}(\xi)|}{(n + 1)!} \cdot h^{n+1}.$$
Proof of Taylor Remainder Theorem

- Intuitively the error of an approximation that takes into account \( n \) derivatives should be proportional to the maximum value of the \( (n + 1) \)th one...

In-class activity: Taylor series
Using Polynomial Approximation

- Suppose we can approximate a function as a polynomial:

  \[ f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3. \]

  How is that useful? Say, if I wanted the integral of \( f \)?

**Demo:** Computing \( \pi \) with Taylor