Overview

- Taylor error
- App. of poly
- Interpolation (give me a poly from f values)
Errors in Taylor Approximation (I)

Can’t sum infinitely many terms. Have to truncate. How big of an error does this cause?

Demo: Polynomial Approximation with Derivatives (Part II)
Taylor Remainders: the Full Truth

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be $(n + 1)$-times differentiable on the interval $(x_0, x)$ with $f^{(n)}$ continuous on $[x_0, x]$. Then there exists a $\xi \in (x_0, x)$ so that

$$f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i = \frac{f^{(n+1)}(\xi)}{(n + 1)!} (\xi - x_0)^{n+1}$$

and since $|\xi - x_0| \leq h$

$$\left| f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i \right| \leq \frac{|f^{(n+1)}(\xi)|}{(n + 1)!} \cdot h^{n+1}.$$
Intuition for Taylor Remainder Theorem

Given the value of a function and its derivative $f(x_0), f'(x_0)$, prove the Taylor error bound.

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(w) \, dw$$

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(w) \, dw + \int_{x_0}^{x} \int_{w_0}^{w} f''(w_1) \, dw_1 \, dw_0$$

In-class activity: Taylor series
\[ f'(x) - f'(x_0) + f''(x_0) \frac{(x-x_0)}{2} \leq \max_{3c > x > 1} |f'''(y)| \cdot \frac{(x-x_0)^2}{2} \]

\[ \text{Error} \propto C \cdot h^{n+1} \]

\[ C \cdot h^n \leq \]
Using Polynomial Approximation

Suppose we can approximate a function as a polynomial:

\[ f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3. \]

How is that useful?
E.g.: What if we want the integral of \( f \)?

\[
\int_0^1 p(x) \, dx \approx \int_0^1 f(x) \, dx = \int_0^1 \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 \right) \, dx
\]

\[
= \left[ a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \frac{a_3 x^4}{4} \right]_0^1
\]

Demo: Computing \( \pi \) with Taylor
$y = \sqrt{1 - x^2}$

$x^2 + y^2 = 1$
Reconstructing a Function From Point Values

If we know function values at some points \( f(x_1), f(x_2), \ldots, f(x_n) \), can we reconstruct the function as a polynomial?

\[
f(x) = a_0 + a_1 x + a_2 x^2 + \cdots
\]

Called "Interpolation" \( p(x) \)

\[
x_1 \rightarrow p(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 + \cdots = f(x_1)
\]

\[
x_2 \rightarrow p(x_2) = a_0 + a_1 x_2 + a_2 x_2^2 + \cdots = f(x_2)
\]

\[
x_n \rightarrow p(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + \cdots = f(x_n)
\]

\( \iff A x = b \)
\[
\begin{pmatrix}
1 \\
1 \\
1 \\
\vdots \\
1 \\
\end{pmatrix}
\begin{pmatrix}
x_1^m \\
x_2^m \\
x_3^m \\
\vdots \\
x_n^m \\
\end{pmatrix}
= 
\begin{pmatrix}
\lambda_0 \\
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_{m-1} \\
\end{pmatrix}
\begin{pmatrix}
\phi(x_1) \\
\phi(x_2) \\
\phi(x_3) \\
\vdots \\
\phi(x_n) \\
\end{pmatrix}
\]

Guess: pick \( n-m \) Vandermonde matrices
Polynomial interpolation is a critical component in many numerical models.

**Demo:** Polynomial Approximation with Point Values
How did the interpolation error behave in the demo?
To fix notation: $f$ is the function we’re interpolating. $\tilde{f}$ is the interpolant that obeys $\tilde{f}(x_i) = f(x_i)$ for $x_i = x_1 < \ldots < x_n$. $h = x_n - x_1$ is the interval length.

What is the error at the interpolation nodes?

Care to make an unfounded prediction? What will you call it?
Let us consider an interpolant $\tilde{f}$ based on $n = 2$ points so

$$\tilde{f}(x_1) = f(x_1) \quad \text{and} \quad \tilde{f}(x_2) = f(x_2).$$

Prove the interpolation error bound in this case.