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Linear Algebra Meets Computation

R. Beeson

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Department of Aerospace Engineering
University of Illinois at Urbana-Champaign

focus of the day

- why linear algebra and computation?
- how do we represent data in a linear algebra form?
- build a connection between data and vectors
- investigate operations on vectors (i.e. data)
- look ahead toward more sophisticated *operators* on data

why linear algebra?

- what connection does linear algebra have with numerics?
- what math operations can we perform on a computer? (think FPU).
- linear algebra. specifically:
  - vectors $\rightarrow$ data
  - matrices $\rightarrow$ *operators* on data

vector addition and subtraction

Addition and subtraction are element-by-element operations

$$c = a + b \iff c_i = a_i + b_i \quad i = 1, \dots, n$$
$$d = a - b \iff d_i = a_i - b_i \quad i = 1, \dots, n$$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$a + b = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \qquad a - b = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

multiplication by a scalar

Multiplication by a scalar involves multiplying each element in the vector by the scalar:

$$b = \sigma a \quad \iff \quad b_i = \sigma a_i \quad i = 1, \ldots, n$$

$$a = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \qquad b = \frac{a}{2} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

linear combinations

Combine scalar multiplication with addition

$$\alpha \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \beta \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \\ \vdots \\ \alpha u_m + \beta v_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

$$r = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \qquad s = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$t = 2r + 3s = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 15 \end{bmatrix}$$

linear combinations

Any one vector can be created from an infinite combination of other "suitable" vectors.

$$w = \begin{bmatrix}4\\2\end{bmatrix} = 4\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$$

$$w = 6\begin{bmatrix}1\\0\end{bmatrix} - 2\begin{bmatrix}1\\-1\end{bmatrix}$$

$$w = \begin{bmatrix}2\\4\end{bmatrix} - 2\begin{bmatrix}-1\\1\end{bmatrix}$$

$$w = 2\begin{bmatrix}4\\2\end{bmatrix} - 4\begin{bmatrix}1\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\end{bmatrix}$$

vector operations

why do these operations make sense?

linear independence and a basis

- A set of vectors $\{u_1, u_2, \ldots, u_m\}$ are said to be **linearly independent** if

$$\sum_{i=1}^{m} \alpha_i u_i = 0 \text{ only when } \alpha_i = 0 \quad \forall i$$

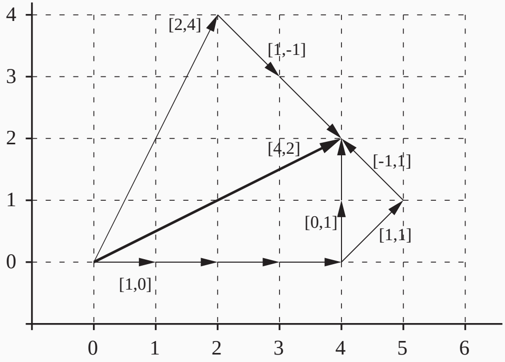
  Otherwise the set is **linearly dependent**.

- A **basis** is a set of linearly independent vectors, such that any other vector is a linear combination of the basis vectors.

linear combinations

Graphical interpretation:

- Vector tails can be moved to convenient locations
- Magnitude and direction of vectors is preserved



linear, affine, and convex combinations

linear:

$$\sum_{i=1}^{n} \alpha_i u_i \quad \alpha_i \in \mathbb{R} \quad u_i \in \mathbb{R}^m$$

affine:

same as linear with the added constraint: $\sum_{i=1}^{n} \alpha_i = 1$

convex:

same as affine with the added constraint: $\alpha_i > 0 \quad \forall i$

vector transpose

The *transpose* of a row vector is a column vector:

$$u = \begin{bmatrix}1, 2, 3\end{bmatrix} \qquad \text{then} \qquad u^T = \begin{bmatrix}1 \\ 2 \\ 3\end{bmatrix}$$

Likewise if v is the column vector

$$v = \begin{bmatrix}4 \\ 5 \\ 6\end{bmatrix} \qquad \text{then} \qquad v^T = \begin{bmatrix}4, 5, 6\end{bmatrix}$$

vector inner product

In physics, analytical geometry, and engineering, the **dot product** has a geometric interpretation

$$\sigma = x \cdot y \quad \Longleftrightarrow \quad \sigma = \sum_{i=1}^{n} x_i y_i$$

$$x \cdot y = \|x\|_2 \, \|y\|_2 \cos \theta$$

vector inner product

The inner product of x and y *requires* that x be a row vector y be a column vector

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$

vector inner product

For two *n*-element *column* vectors, u and v, the inner product is

$$\sigma = u^T v \quad \Longleftrightarrow \quad \sigma = \sum_{i=1}^{n} u_i v_i$$

The inner product is commutative so that
(for two column vectors)

$$u^T v = v^T u$$

vector outer product

The inner product results in a scalar.

The *outer product* creates a rank-one matrix:

$$A = uv^T \iff a_{i,j} = u_i v_j$$

operators on vectors (i.e. data)

Vectors (i.e. data) is one-half of our Linear Algebra. The other focuses on *Operators* acting on the vectors.
What can these operators do?

- Scaling
- Permutations
- Rotation
- Used in Linear System Solves

Do they (*Operators*) have another name?

- Matrices

notation

The operator A with m rows and n columns looks like:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} = element in **row** i, and **column** j

Recall the significance of the entry position for a vector (e.g. if $a = [1, 2, 4, 9, \ldots]$, what is the meaning of 1 in the first entry). What is the significance of the entry positions in A then?

matrices consist of row and column vectors

As a collection of column vectors

$$A = \left[\begin{array}{c|c|c|c} & & & \\ a_{(1)} & a_{(2)} & \cdots & a_{(n)} \\ & & & \end{array}\right]$$

As a collection of row vectors

$$A = \left[\begin{array}{c} a'_{(1)} \\ \hline a'_{(2)} \\ \hline \vdots \\ \hline a'_{(m)} \end{array}\right]$$

some remarks

- data can be represented by vectors in the linear algebra sense
- we have seen how to perform vector operations
- we will see *operators* can be applied to the data to yield more interesting and useful results
- linear algebra forms the base of our numerical methods