A Note on Convergence Rate for Newton and Secant Methods

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1 Fixed Point Iteration

We are interested in the error behavior of nonlinear iteration schemes. If x^* is our solution and x_k the current guess, then the error is $e_k := x_k - x^*$. If we have

$$\lim_{k \longrightarrow \infty} \frac{|e_{k+1}|}{|e_k|^r} \quad = \quad C,$$

we say that the convergence is of order r.

Consider a fixed-point iteration, $x_{k+1} = g(x_k)$. Using a Taylor series expansion about x^*, there exists $\theta_k \in [x_k, x^*]$ such that

$$x_{k+1} \quad = \quad g(x_k) = g(x^*) + e_k g'(\theta_k).$$

Subtracting $x^* = g(x^*)$ from both sides, we have

$$e_{k+1} \quad = \quad e_k g'(\theta_k),$$

or

$$\frac{e_{k+1}}{e_k} \quad = \quad g'(\theta_k).$$

If $g'(x^*) \neq 0$ then the order of convergence is $r = 1$ and $C = g'(x^*)$.

If $g'(x^*) = 0$, we take a two term Taylor series expansion,

$$x_{k+1} \quad = \quad g(x_k) = g(x^*) + e_k g'(x^*) + \frac{e_k^2}{2} g''(xi_k),$$

from which we find

$$\frac{e_{k+1}}{e_k^2} \quad = \quad g''(\xi_k),$$

for some $\xi_k \in [x_k, x^*]$.

2 Secant Method

Newton's method is a proper fixed point interation of the form $x_{k+1} = g(x_k)$ with $g'(x^*) = 0$ (unless there is a multiplicity of order $m > 1$ at x^*).

By contrast, the *secant method* is of the form $x_{k+1} = g(x_k, x_{k-1})$. Its error behavior is slightly different and we'll need to look at its precise formulation in some detail to arrive at the order of convergence.

Recall Newton's method

$$x_{k+1} \;=\; x_k - \frac{1}{f'_k}\, f_k.$$

For the secant method, we approximate f'_k as

$$f'_k \;\approx\; \frac{f_k - f_{k-1}}{x_k - x_{k-1}},$$

which leads to

$$x_{k+1} \;=\; x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}}\, f_k.$$

Subtracting x^* from both sides we have the error equation

$$\begin{aligned}
e_{k+1} &= e_k - \frac{e_k - e_{k-1}}{f_k - f_{k-1}}\, f_k \\
&= \frac{f_k e_k - f_{k-1} e_k}{f_k - f_{k-1}} - \frac{f_k e_k - f_k e_{k-1}}{f_k - f_{k-1}} \\
&= \frac{f_k e_{k-1} - f_{k-1} e_k}{f_k - f_{k-1}} \\
&= \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right)\left(\frac{f_k e_{k-1} - f_{k-1} e_k}{x_k - x_{k-1}}\right). \\
&= \left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right)\left(\frac{f_k/e_k - f_{k-1}/e_{k-1}}{x_k - x_{k-1}}\right) e_k\, e_{k-1}
\end{aligned}$$

If the scheme is convergent, the first term approaches $1/f'(x^*)$. For the second term, use a Taylor series about x^* and the fact that $f(x^*) = 0$ to note that

$$\frac{f_k}{e_k} \;=\; \frac{f_k - f(x^*)}{x_k - x^*} \;=\; f'(x^*) \;+\; \frac{e_k}{2} f''(x^*) + \text{h.o.t.}.$$

Thus

$$\left(\frac{f_k/e_k - f_{k-1}/e_{k-1}}{x_k - x_{k-1}}\right) \;\approx\; \left(\frac{\frac{1}{2} e_k f'' - \frac{1}{2} e_{k-1} f''}{e_k - e_{k-1}}\right) \;\approx\; \frac{1}{2} f''(x^*).$$

Combining the results, we have

$$\begin{aligned}
\lim_{k \longrightarrow \infty} e_{k+1} &= \frac{1}{2}\frac{f''(x^*)}{f'(x^*)}\, e_k\, e_{k-1}\,. \\
&= A\, e_k\, e_{k-1}\,. \qquad (1)
\end{aligned}$$

The next step is to determine the rate of convergence, r.

To do so, we use the rate of convergence ansatz,

$$|e_{k+1}| \quad \sim \quad C|e_k|^r, \tag{2}$$

from which we also have

$$|e_k| \quad \sim \quad C|e_{k-1}|^r, \tag{3}$$

$$|e_{k-1}| \quad \sim \quad ((|e_k|/C)^{1/r}\,. \tag{4}$$

Using (1) and (2), we have

$$C|e_k|^r \sim |e_{k+1}| \quad \sim \quad A|e_k|\,|e_{k-1}| \tag{5}$$

$$\sim \quad A|e_k|\,((|e_k|/C)^{1/r}\,. \tag{6}$$

$$\sim \quad A|e_k|^{1+\frac{1}{r}}C^{-\frac{1}{r}}. \tag{7}$$

Consolidating,

$$C^{1+\frac{1}{r}}A^{-1} \quad \sim \quad |e_k|^{1+\frac{1}{r}-r} \; = \; \text{constant as } k \longrightarrow \infty. \tag{8}$$

Since the left-hand side is a constant (in the limit), then we must have

$$1 + \frac{1}{r} - r = 0$$

or

$$r^2 - r - 1 = 0,$$

which has the solution

$$r \quad = \quad \frac{1+\sqrt{5}}{2} \approx 1.62.$$