Finite Difference Methods

Let's derive the finite difference method for the ODE BVP defined by

\[ u'' + 1000(1 + t^2)u = 0 \]

with boundary conditions \( u(-1) = 3 \) and \( u(1) = -3 \).

Using a discretization with points \( t_1, \ldots, t_n, \) \( t_{i+1} - t_i = h, \) and a centered difference approximation for \( u'' \) we obtain

\[ \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + 1000(1 + t_i)u_i = 0. \]

We can rewrite the above using linear equations with matrices

\[
\begin{bmatrix}
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\cdot & \cdot & \cdot \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\vdots \\
u_n \\
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}
\begin{bmatrix}
1000(1 + t_2) \\
1000(1 + t_{n-1}) \\
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\vdots \\
u_n \\
\end{bmatrix}
\end{align*}

and solve the system \((A + B)u = \begin{bmatrix} 3 & 0 & \cdots & 0 & -3 \end{bmatrix}^T.\)
Collocation Methods

- **Collocation methods** approximate $y$ by representing it in a basis

\[ y(t) = v(t, x) = \sum_{i=1}^{n} x_i \phi_i(t). \]

To construct equations, consider approximation for a set of collocation points $t_1, \ldots, t_n$ with $t_1 = a$ and $t_n = b$,

\[ \forall i \in \{2, \ldots, n-1\} \quad v(t_i, x) = f(t_i, v(t_i, x)), \]

with two more equations typically obtained from boundary conditions at $t_1, t_n$.

- **Spectral methods** use polynomials or trigonometric functions for $\phi_i$, which are nonzero over most of $[a, b]$, while **finite element** methods leverage basis functions with local support (e.g. B-splines).

Eigenfunctions of differential operators are typically trigonometric functions or polynomials, hence the name “spectral methods”.
We reformulate the collocation approximation as an optimization problem:

Consider the simplified scenario \( f(t, y) = f(t) \) with residual equation,

\[
\begin{align*}
    r(t, x) &= v'(t, x) - f(t) = \sum_{j=1}^{n} x_j \phi_j'(t) - f(t) \\
    \text{and minimize it using the objective function,}
    \quad F(x) &= \frac{1}{2} \int_{a}^{b} ||r(t, x)||_2^2 dt.
\end{align*}
\]

The first-order optimality conditions of the optimization problem are a system of linear equations \( A x = b \):

\[
\begin{align*}
    0 &= \frac{dF}{dx_i} = \int_{a}^{b} r(t, x)^T \frac{dr}{dx_i} dt = \int_{a}^{b} r(t, x)^T \phi_i'(t) dt \\
    &= \sum_{j=1}^{n} x_j \int_{a}^{b} \phi_j'(t)^T \phi_i'(t) dt - \int_{a}^{b} f(t)^T \phi_i'(t) dt
\end{align*}
\]
Weighted Residual

- **Weighted residual methods** work by ensuring the residual is orthogonal with respect to a given set of weight functions:

  *Rather than setting components of the gradient to zero, we instead have,*

  \[
  \int_a^b r(t, \mathbf{x})^T \mathbf{w}_i(t) dt = 0, \forall i \in \{1, \ldots, n\},
  \]

  *which again yields a system of equations of the form \( Ax = b \)* where

  \[
  a_{ij} = \int_a^b \phi'_j(t)^T \mathbf{w}_i(t), \quad b_i = \int_a^b \mathbf{f}(t)^T \mathbf{w}_i(t).
  \]

  *The collocation method is a weighted residual method where \( \mathbf{w}_i(t) = \delta(t - t_i) \).*

  *The Galerkin method is a weighted residual method where \( \mathbf{w}_i = \phi_i \).*

  *Linear system with the stiffness matrix \( A \) and load vector \( b \) is*

  \[
  0 = \sum_{j=1}^n x_j \int_a^b \phi'_j(t)^T \phi_i(t) dt - \int_a^b \mathbf{f}(t)^T \phi_i(t) dt.
  \]
Linear BVPs by Optimization

- Let's apply the Galerkin method to the more general linear ODE \( f(t, y) = A(t)y(t) + b(t) \) with residual equation,

  First, choose basis functions \( \{\phi_i\}_{i=1}^{n} \) to satisfy the boundary conditions, so the solution automatically satisfies them, then minimize the residual,

  \[
  r = v' - Av - b, \quad \text{so that} \quad r(t, x) = \sum_{j=1}^{n} x_j (\phi'_j(t) - A(t)\phi_j(t)) - b(t).
  \]

  The Galerkin method, minimizes the residual by orthogonality with respect to a set of test functions that is the same as the set of basis functions,

  \[
  0 = \int_{a}^{b} r(t, x)^{T} \phi_i(t) dt = \sum_{j=1}^{n} x_j \int_{a}^{b} (\phi'_j(t) - A(t)\phi_j(t))^{T} \phi_i(t) dt - \int_{a}^{b} b(t)^{T} \phi_i(t) dt
  \]
Nonlinear BVPs: Poisson Equation

In practice, BVPs are at least second order and it's advantageous to work in the natural set of variables.

- Consider the Poisson equation $u'' = f(t)$ with boundary conditions $u(a) = u(b) = 0$ and define a localized basis of hat functions:

  \[
  \phi_i(t) = \begin{cases} 
  \frac{(t - t_{i-1})}{h} & : t \in [t_{i-1}, t_i] \\
  \frac{(t_{i+1} - t)}{h} & : t \in [t_i, t_{i+1}] \\
  0 & : \text{otherwise}
  \end{cases}
  \]

  where $t_0 = t_1 = a$ and $t_{n+1} = t_n = b$.

  Trying to define the residual equation as usual, we obtain

  \[
  r = v'' - f, \quad \text{so that} \quad r(t, \mathbf{x}) = \sum_{j=1}^{n} x_j \phi''_j(t) - f(t).
  \]

  However, $\phi''_j(t)$ is undefined, since $\phi'_j(t)$ is discontinuous at $t_{j-1}, t_j, t_{j+1}$. 
Weak Form and the Finite Element Method

The finite-element method permits a lesser degree of differentiability of basis functions by casting the ODE in weak form:

For any solution \( u \), if test function \( \phi_i \) satisfies the boundary conditions, the ODE satisfies the weak form,

\[
\int_{a}^{b} f(t) \phi_i(t) dt = \int_{a}^{b} u''(t) \phi_i(t) dt = u'(b) \phi_i(b) - u'(a) \phi_i(a) - \int_{a}^{b} u'(t) \phi'_i(t) dt \\
= - \int_{a}^{b} u'(t) \phi'_i(t) dt.
\]

Note that the final equation contains no second derivatives, and subsequently we can form the linear system \( Ax = b \) with,

\[
a_{ij} = - \int_{a}^{b} \phi'_j(t) \phi'_i(t) dt, \quad b_i = \int_{a}^{b} f(t) \phi_i(t) dt.
\]

The finite element method thus searches the larger (once-differentiable) function space to find a solution \( u \) that is in a (twice-differentiable) subspace.
Hat functions are linear instances of \textit{B-splines}:
\begin{itemize}
\item[$\triangleright$] \textit{B-splines of degree $k$ are $k$-times differentiable. For higher-order ODEs or high-order convergence with $h$, its necessary to use $k > 1$.}
\end{itemize}

Finite-element methods readily generalize to PDEs:
\begin{itemize}
\item[$\triangleright$] \textit{In its most basic form each element corresponds to a triangle (2D) or quadrilateral (3D).}
\end{itemize}
A typical second-order scalar BVP eigenvalue problem has the form

\[ u'' = \lambda f(t, u, u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0 \]

Lets first consider \( f(t, u, u') = g(t)u \), in which case we can approximate the solution at a set of points \( t_1, \ldots, t_n \) using finite differences,

\[ \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \lambda g_i y_i, \]

which corresponds to a tridiagonal matrix eigenvalue problem \( A y = \lambda y \) via

\[ \frac{y_{i+1} - 2y_i + y_{i-1}}{g_i h^2} = \lambda y_i. \]
Eigenvalue Problems with ODEs

Generalized eigenvalue problems arise from more sophisticated ODEs,

\[ u'' = \lambda (g(t)u + h(t)u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0 \]

We can approximate each of the derivatives at a set of points \( t_1, \ldots, t_n \) using finite differences,

\[
\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \lambda \left( g_i + \frac{y_{i+1} - y_{i-1}}{2h} \right) y_i.
\]

which can be expressed as a generalized matrix eigenvalue problem

\[ Ay = \lambda By \]

where both \( A \) and \( B \) are tridiagonal.