CS 450: Numerical Analysis

Boundary Value Problems for Ordinary Differential Equations

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These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book “Scientific Computing: An Introductory Survey” by Michael T. Heath (slides).
Boundary Conditions

- Often we seek to solve a differential equation that satisfies conditions on its values and derivatives on parts of the domain boundary.

Consider a first order ODE $y'(t) = f(t, y)$ with linear boundary conditions on domain $t \in [a, b]$:

$$B_a y(a) + B_b y(b) = c$$

If we can decouple $y(a)$ and $y(b)$, these are separated.
Existence of Solutions for Linear ODE BVPs

- The solutions of linear ODE BVP \( y'(t) = A(t)y(t) + b(t) \) are linear combinations of solutions to linear homogeneous ODE IVPs:

\[
\begin{align*}
  y'(t) &= A(t)y(t) \\
  y(0) &= c
\end{align*}
\]

Let \( y_i(t), \ a \neq 0, \ y_i(0) = e_i \), as columns of

\[
Y(t) = I + \int_a^t A(s)Y(s)\, ds, \quad y(t) = Y(t)u(t)
\]

\[
Y'(t)u(t) + Y(t)u'(t) = A(t)Y(t)u(t) + b(t)
\]

- Solution \( u(t) \) and \( y(t) \) exist if \( Q = B_aY(a) + B_bY(b) \) is invertible:

\[
\begin{align*}
  u(t) &= u(a) + \int_a^t u'(s)\, ds \\
  y(t) &= \left( B_aY(a) + B_bY(b) \right)^{-1}(c - B_bY(b)\int_a^t u'(s)\, ds)
\end{align*}
\]
Green’s Function Form of Solution for Linear ODE BVPs

For any given \( b(t) \) and \( c \), the solution to the BVP can be written in the form:

\[
y(t) = \Phi(t) c + \int_a^b G(t, s) b(s) \, ds
\]

\( \Phi(t) = Y(t)Q^{-1} \) is the fundamental matrix and the Green’s function is

\[
G(t, s) = Y(t)Q^{-1} I(s) Y^{-1}(s) \quad I(s) = \begin{cases} 
B_a Y(a) & : s < t \\
-B_b Y(b) & : s \geq t
\end{cases}
\]

\[
y(t) = Y(t) c + \int_a^b G(t, s) b(s) \, ds
\]

\[
= Y(t) \left( Q^{-1} c - Q^{-1} B_b Y(b) \int_a^b u'(s) \, ds \right) + \int_a^b \left( B_a Y(a) \int_a^b u'(s) \, ds - B_b Y(b) \int_a^b u'(s) \, ds \right) b(s) \, ds
\]

\[
= \Phi(t) c + Y(t) \left( Q^{-1} c - Q^{-1} B_b Y(b) \int_a^b u'(s) \, ds \right) + \int_a^b \left( B_a Y(a) \int_a^b u'(s) \, ds - B_b Y(b) \int_a^b u'(s) \, ds \right) b(s) \, ds
\]

\[
= \Phi(t) c + Y(t) Q^{-1} \left( B_a Y(a) \int_a^b u'(s) \, ds - B_b Y(b) \int_a^b u'(s) \, ds \right)
\]
For any given $b(t)$ and $c$, the solution to the BVP can be written in the form:

$$y(t) = \Phi(t)c + \int_{a}^{b} G(t, s)b(s) \, ds$$

The absolute condition number of the BVP is $\kappa = \max\{\|\Phi\|_\infty, \|G\|_\infty\}$.
Shooting Method for ODE BVPs

- For linear ODEs, we construct solutions from IVP solutions in $Y(t)$, which suggests the *shooting method* for solving BVPs by reduction to IVPs:

1. guess an I.C., so $y^{(k)}(a)$
2. solve IVP w. I.C. $y^{(k)}(a)$ to obtain $y^{(k)}(b)$
3. check distance from satisfying BCs,
   \[ ||B_a y^{(c)} + B_b y(b) - c || \]
4. so Pick $y^{(k+1)}(a)$ by trying $y^{(k)}(a)$ as guesses
   \( x_1, x_2, \ldots, x_k \) to root finders for

- *Multiple shooting* employs the shooting method over subdomains:

IVP from single-shooting is often worse-conditioned than BVP, so may be unstable.

This guess system of nonlinear eqs.
Finite Difference Methods

- Rather than solve a sequence of IVPs that satisfy the ODEs until they (approximately) satisfy boundary conditions, we can refine an approximation that satisfies the boundary conditions, until it satisfies the ODE:

\[
\text{Convergence to solution is obtained with decreasing step size } h \text{ so long as the method is consistent and stable:}
\]
Let's derive the finite difference method for the ODE BVP defined by

\[ u'' = -1000(1 + t^2)u \]

\[ u'' + 1000(1 + t^2)u = 0 \]

with boundary conditions \( u(-1) = 3 \) and \( u(1) = -3 \), using a centered difference approximation for \( u'' \) on \( t_1, \ldots, t_n, t_{i+1} - t_i = h \).

\[
\begin{align*}
\frac{1}{h^2} (u_{i+1} - 2u_i + u_{i-1}) &= -1000 \left( 1 + t_i^2 \right) u_i \\
\end{align*}
\]

\[
\begin{pmatrix}
\frac{1}{h^2} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & \frac{1}{h^2} & 0 \\
0 & \cdots & 0 & \frac{1}{h^2}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}
\begin{pmatrix}
-1000 \left( 1 + t_1^2 \right) \\
\vdots \\
-1000 \left( 1 + t_n^2 \right)
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{h^2} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & \frac{1}{h^2} & 0 \\
0 & \cdots & 0 & \frac{1}{h^2}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}
\begin{pmatrix}
-1000 \left( 1 + t_1^2 \right) \\
\vdots \\
-1000 \left( 1 + t_n^2 \right)
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
u_1 = 3 \\
u_n = -3
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}
\begin{pmatrix}
1 + (n-1)^2 \\
\vdots \\
1 + (-1)^2
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
3 \\
0 \\
\vdots \\
0 \\
-3
\end{pmatrix}
\]

\[
(A + B)u = \begin{bmatrix} 3 & 0 & \cdots & 0 & -3 \end{bmatrix}^T
\]
Collocation Methods

- **Collocation methods** approximate $y$ by representing it in a basis

$$y(t) \approx v(t, x) = \sum_{i=1}^{n} x_i \phi_i(t).$$

- Choices of basis functions give different families of methods:

  - Polynomials or Lagrange basis functions. Good because they are eigenfunctions of differential operators.
  - Special but because they are nonzero everywhere.
  - FEM: B-splines, or localized functions. Sparse linear systems.
Solving BVPs by Optimization

- Can solve collocation equations by minimizing residual (for simplified scenario $f(t, y) = f(t)$),

$$ r(t, x) = v'(t, x) - f(t) = \sum_{j=1}^{n} x_j \phi'_j(t) - f(t). $$

- The first-order optimality conditions of the optimization problem are a system of linear equations $Ax = b$:

$$ \frac{\partial F}{\partial x_i} = \int_{a}^{b} \epsilon'_i(s) \cdot r(s, x) \, ds \Rightarrow \sum_{i=1}^{n} x_i \int_{0}^{b} \epsilon'_i(s) \cdot \phi'_i(t) \, ds = \int_{0}^{b} f(s) \, ds $$
Weighted Residual

- **Weighted residual methods** work by ensuring the residual is orthogonal with respect to a given set of weight functions:

\[
\int w_i(t) r(t, x) \, dt = 0 \quad \forall \ i \ - \ weight \ function
\]

- The Galerkin method is a weighted residual method where \( w_i = \phi_i \).
Second-Order BVPs: Poisson Equation

In practice, BVPs are at least second order and it's advantageous to work in the natural set of variables.

Consider the Poisson equation \( u'' = f(t) \) with boundary conditions \( u(a) = u(b) = 0 \) and define a localized basis of hat functions:

\[
\phi_i(t) = \begin{cases} 
  (t - t_{i-1})/h & : t \in [t_{i-1}, t_i] \\
  (t_{i+1} - t)/h & : t \in [t_i, t_{i+1}] \\
  0 & : \text{otherwise}
\end{cases}
\]

where \( t_0 = t_1 = a \) and \( t_{n+1} = t_n = b \).
Weak Form and the Finite Element Method

- The finite-element method permits a lesser degree of differentiability of basis functions by casting the ODE in *weak form*:
Eigenvalue Problems with ODEs

A typical second-order scalar *ODE BVP eigenvalue problem* is

\[ u'' = \lambda f(t, u, u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0. \]

These can be solved, e.g. for \( f(t, u, u') = g(t)u \) by finite differences:
Using Generalized Matrix Eigenvalue Problems

- Generalized matrix eigenvalue problems arise from more sophisticated ODEs,

\[ u'' = \lambda (g(t)u + h(t)u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0. \]