CS 450: Numerical Anlaysis[1]

Boundary Value Problems for Ordinary Differential Equations

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[1] *These slides have been drafted by Edgar Solomonik as lecture templates and supplementary material for the book "Scientific Computing: An Introductory Survey" by Michael T. Heath (slides).*

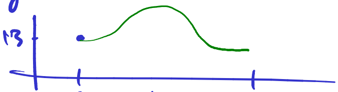
Boundary Conditions

- Often we seek to solve a differential equation that satisfies conditions on its values and derivatives on parts of the domain boundary.

BC: Dirichlet

Neumann

$y(a) = 13$



$y'(b) = 0$

or

$y(b) = 17$

$y'(t) = f(t, y)$

differentiating t

$y(t)$

- Consider a first order ODE $\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y})$ with *linear boundary conditions* on domain $t \in [a, b]$:

initial point, final point

$$\boldsymbol{B}_a \boldsymbol{y}(a) + \boldsymbol{B}_b \boldsymbol{y}(b) = \boldsymbol{c}$$

$$\begin{bmatrix} B_a & B_b \end{bmatrix} \begin{bmatrix} y(a) \\ y(b) \end{bmatrix} = c$$

these are separated, if we can decouple $y(a)$ and $y(b)$

Existence of Solutions for Linear ODE BVPs

▸ The solutions of linear ODE BVP $\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t) + \boldsymbol{b}(t)$ are linear combinations of solutions to linear homogeneous ODE IVPs $\boxed{\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t)}$:

no $b(t)$

Let $y_i(t)$, at $t=0$, $y_i(0) = e_i$, as columns of

$$Y(t) = I + \int_a^t \underbrace{A(s)Y(s)}_{Y'(s)}\,ds, \quad \underset{\text{soln to BVP}}{y(t)} = Y(t)\underbrace{u(t)}_{\text{unknown that depends on } b(t) \text{ and BCs}}$$

$$\cancel{Y'(t)u(t)} + Y(t)u'(t) = \cancel{A(t)Y(t)u(t)} + b(t)$$

$$u'(t) = Y(t)^{-1}b(t)$$

▸ Solution $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$ exist if $\boldsymbol{Q} = \boldsymbol{B}_a\boldsymbol{Y}(a) + \boldsymbol{B}_b\boldsymbol{Y}(b)$ is invertible:

$$u(b) = u(a) + \int_a^b u'(s)\,ds, \quad \text{so BCs entail}$$

$$B_a \underbrace{Y(a)u(a)}_{y(a)} + B_b Y(b)\left(u(a) + \int_a^b u'(s)\,ds\right) = c$$

$$u(a) = \underbrace{\left(B_a Y(a) + B_b Y(b)\right)}_{Q}^{-1}\left(c - B_b Y(b)\int_a^b u'(s)\,ds\right)$$

Green's Function Form of Solution for Linear ODE BVPs

- For any given $\boldsymbol{b}(t)$ and $\boldsymbol{c}$, the solution to the BVP can be written in the form:

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c} + \int_a^b \boldsymbol{G}(t,s)\boldsymbol{b}(s)\,ds \qquad Q = B_aY(a) + B_bY(b)$$

$\boldsymbol{\Phi}(t) = \boldsymbol{Y}(t)\boldsymbol{Q}^{-1}$ is the *fundamental matrix* and the *Green's function* is

$$\boldsymbol{G}(t,s) = \boldsymbol{Y}(t)\boldsymbol{Q}^{-1}\boldsymbol{I}(s)\boldsymbol{Y}^{-1}(s), \qquad \boldsymbol{I}(s) = \begin{cases} \boldsymbol{B}_a\boldsymbol{Y}(a) & : s < t \quad [a,t] \\ -\boldsymbol{B}_b\boldsymbol{Y}(b) & : s \geq t \quad [t,b] \end{cases}$$

$$y(t) = Y(t)\,u(t) = Y(t)\left(u(a) + \int_a^t u'(s)\,ds\right)$$

$$= Y(t)\left(Q^{-1}c - Q^{-1}B_bY(b)\int_a^b u'(s)\,ds + \int_a^t u'(s)\,ds\right)$$

$$= \Phi(t)c + Y(t)Q^{-1}\left(-B_bY(b)\int_a^b \underbrace{u'(s)}_{Y^{-1}(s)b(s)}\,ds + Q\int_a^t u'(s)\,ds\right)$$

$$= \Phi(t)c + Y(t)Q^{-1}\left(B_aY(a)\int_a^t Y^{-1}(s)b(s)\,ds - B_bY(b)\int_t^b Y^{-1}(s)b(s)\,ds\right)$$

Conditioning of Linear ODE BVPs

- For any given $\boldsymbol{b}(t)$ and $\boldsymbol{c}$, the solution to the BVP can be written in the form:

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c} + \int_a^b \boldsymbol{G}(t,s)\boldsymbol{b}(s)ds$$

$\varphi(t)$ is also independent of $b(t)$

$\to G$ is independent of $b(s)$, but dependent on BCs, it can be expressed in terms of φ and ψ

- The absolute condition number of the BVP is $\kappa = \max\{||\boldsymbol{\Phi}||_\infty, ||\boldsymbol{G}||_\infty\}$:

Demo: *Shooting method*

Shooting Method for ODE BVPs

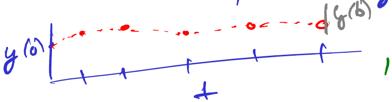
- For linear ODEs, we construct solutions from IVP solutions in $\boldsymbol{Y}(t)$, which suggests the *shooting method* for solving BVPs by reduction to IVPs:

1. guess an I.C., so $y^{(k)}(a)$
2. solve IVP with I.C. $y^{(k)}(a)$ to obtain $y^{(k)}(b)$
3. check distance from satisfying BCs, $\| B_a y(a) + B_b y(b) - c \|$
4. so pick $\underbrace{y^{(k+1)}(a)}_{\text{I.C.}}$ by treating $y^{(1)}(a) \dots y^{(k)}(a)$ as guesses $x_1 \dots x_k$ to root finder for

$$h(x) = B_a x + B_b y_x - c$$
where $y_{y^{(1)}(a)} = y^{(1)}(b)$

- *Multiple shooting* employs the shooting method over subdomains:

IVP from single-shooting is often worse-conditioned than BVP, so may be unstable.



this gives system of nonlinear eqs.

Finite Difference Methods

- Rather than solve a sequence of IVPs that satisfy the ODEs until they (approximately) satisfy boundary conditions, we can refine an approximation that satisfies the boundary conditions, until it satisfies the ODE:

instead can build in BCs into approximation → $t_{i+1} - t_i = h$

introduce points $t_1, \ldots, t_n$, $t_1 = a$, $t_n = b$

approximate solutions $y_1 \ldots y_n$, $y(t_i) \approx y_i$

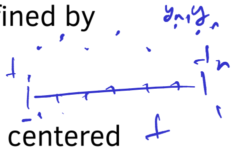
BCs provide 2 eqs for t_1 and t_n

$$\forall i \in [2, n-1], \quad \frac{y_{i+1} - y_{i-1}}{2 \cdot h} = f(t_i, y_i)$$

- Convergence to solution is obtained with decreasing step size h so long as the method is consistent and stable:

Finite Difference Methods

- Lets derive the finite difference method for the ODE BVP defined by



$$u'' = -1000(1+t^2)u$$

$$u'' + 1000(1+t^2)u = 0$$

with boundary conditions $u(-1) = 3$ and $u(1) = -3$, using a centered difference approximation for u'' on $t_1, \dots, t_n$, $t_{i+1} - t_i = h$.

$$\forall i \in 2..n-1 \quad \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = -1000(1+t_i^2)u_i$$

$$u_1 = 3$$
$$u_n = -3$$

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} 0 & \cdots & & 0 \\ 1 & -2 & 1 & \\ & & \ddots & \\ & 1 & -2 & 1 \\ 0 & \cdots & & 0 \end{bmatrix}}_{A} u + \underbrace{1000\begin{bmatrix} 1 & & & \\ & 1+(h-1)^2 & & \\ & & \ddots & \\ & & & 1+(-h)^2 \\ & & & & 1 \end{bmatrix}}_{B} u = \begin{bmatrix} 3 \\ 0 \\ \vdots \\ 0 \\ -3 \end{bmatrix}$$

linear system

$$\boxed{(A+B)} u = [3 \; 0 \cdots 0 \; -3]^T$$

Collocation Methods

- *Collocation methods* approximate $\boldsymbol{y}$ by representing it in a basis

$$\boldsymbol{y}(t) \approx \boldsymbol{v}(t, \boldsymbol{x}) = \sum_{i=1}^{n} x_i \boldsymbol{\phi}_i(t).$$

basis function

$$v'(t_i, x) = f(t_i, v(t_i, x)) \quad \forall\, t_i$$

collocation points

$$\sum_{j=1}^{n} x_j \varphi_j'(t_i) = f\left(t_i, \sum_{j=1}^{n} x_j \varphi_j(t_i)\right)$$

coefficient points ODE

- Choices of basis functions give different families of methods:

  · polynomials or trigonometric fun's

  spectral → · good because they are eigenfunctions of diff'ops

  · bad because they are nonzero everywhere

  FEM → · B-splines, or localized functions → sparse linear systems

Solving BVPs by Optimization

- Can solve collocation equations by minimizing residual (for simplified scenario $\boldsymbol{f}(t,y)=\boldsymbol{f}(t)$),

approx — ODE conditions

$$\boldsymbol{r}(t,\boldsymbol{x})=\boldsymbol{v}'(t,\boldsymbol{x})-\boldsymbol{f}(t)=\sum_{j=1}^{n}x_j\boldsymbol{\phi}_j'(t)-\boldsymbol{f}(t).$$

Minimize objective function

$$F(x)=\frac{1}{2}\int_a^b \|r(s,x)\|_2^2\,ds$$

- The first-order optimality conditions of the optimization problem are a system of linear equations $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$:

$$\forall i \quad 0=\frac{dF}{dx_i}=\int_a^b \varphi_i'(s)\cdot \underbrace{r(s,x)}_{\sum_{j=1}^n x_j\varphi_j'(t)-f(t)}\,ds \Rightarrow \sum_{j=1}^n x_j \overbrace{\int_a^b \varphi_i'(s)\varphi_j'(s)\,ds}^{A_{ij}} = \int_a^b f(s)\varphi_i'(s)\,ds$$

Weighted Residual

- *Weighted residual methods* work by ensuring the residual is orthogonal with respect to a given set of weight functions:

$\{w_i(t)\}_{i=1}^r$ - weight functions, the residual is orthogonal to this basis

$$\int_a^b w_i(t)\, r(t, \underset{\text{coefficients}}{\uparrow x})\, dt = 0 \quad \forall i \text{ - weight functions}$$

- The Galerkin method is a weighted residual method where $w_i = \phi_i$.

$$\int_a^b \varphi_i(t)\, r(t,x)\, dt = 0$$

$$\int_a^b \varphi_i(t) \Big(\overbrace{\sum_{j=1}^n x_j \varphi_j'(t)}^{v'(t,x)} - f(t) \Big)\, dt = 0$$

$$\Rightarrow \sum_{j=1}^n x_j \underbrace{\int_a^b \varphi_i(t)\, \varphi_j'(t)\, dt}_{A_{ij}} = \overbrace{\int_a^b \varphi_i(t) f(t)\, dt}^{b_i}$$

Second-Order BVPs: Poisson Equation

In practice, BVPs are at least second order and its advantageous to work in the natural set of variables.

- Consider the *Poisson equation* $u'' = f(t)$ with boundary conditions $u(a) = u(b) = 0$ and define a localized basis of hat functions:

$$\phi_i(t) = \begin{cases} (t - t_{i-1})/h & : t \in [t_{i-1}, t_i] \\ (t_{i+1} - t)/h & : t \in [t_i, t_{i+1}] \\ 0 & : \text{otherwise} \end{cases}$$

  where $t_0 = t_1 = a$ and $t_{n+1} = t_n = b$.

Weak Form and the Finite Element Method

- The finite-element method permits a lesser degree of differentiability of basis functions by casting the ODE in *weak form*:

Eigenvalue Problems with ODEs

- A typical second-order scalar *ODE BVP eigenvalue problem* is

  $$u'' = \lambda f(t, u, u'), \quad \text{with boundary conditions } u(a) = 0, u(b) = 0.$$

  These can be solved, e.g. for $f(t, u, u') = g(t)u$ by finite differences:

Using Generalized Matrix Eigenvalue Problems

- Generalized matrix eigenvalue problems arise from more sophisticated ODEs,

$$u'' = \lambda(g(t)u + h(t)u'), \quad \text{with boundary conditions } \; u(a) = 0, u(b) = 0.$$