Simpler Problems: Triangular

What do we win from transforming a least-squares system to upper triangular form?

\[ \| Ax - b \|_2 = \| Q^T(Ax - b) \|_2 = \| Q^T(QLx - b) \|_2 = \| Rx - Q^Tb \|_2 \]

\[ \text{min} \]

How would we minimize the residual norm?
Computing QR

- Gram-Schmidt
- Householder Reflectors
- Givens Rotations

**Demo:** Gram-Schmidt–The Movie [cleared]
**Demo:** Gram-Schmidt and Modified Gram-Schmidt [cleared]
**Demo:** Keeping track of coefficients in Gram-Schmidt [cleared]

Seen: Even modified Gram-Schmidt still unsatisfactory in finite precision arithmetic because of roundoff.

**NOTE:** Textbook makes further modification to ‘modified’ Gram-Schmidt:
- Orthogonalize subsequent rather than preceding vectors.
- Numerically: no difference, but sometimes algorithmically helpful.
Is QR with square $Q$ for $A \in \mathbb{R}^{m \times n}$ with $m > n$ efficient?

\[ ||Ax - b||_2 = ||Q(Rx - QTb)||_2 \]

\[ A = \begin{bmatrix} m \end{bmatrix} \quad m \Rightarrow \begin{bmatrix} m \end{bmatrix} \quad n \]
In-Class Activity: QR

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Householder Transformations

Find an *orthogonal* matrix $Q$ to zero out the lower part of a vector $a$. 

\[ a - u ||a\cdot e_1 = v \]

\[ H = I - 2 \frac{vv^T}{v^Tv} \]

Householder reflector.
Householder Reflectors: Properties

Seen from picture (and easy to see with algebra):

\[ Ha = \pm \|a\|_2 \, e_1. \]

Remarks:

- **Q:** What if we want to zero out only the \( i + 1 \)th through \( n \)th entry?
  **A:** Use \( e_i \) above.

- A product \( H_n \cdots H_1 A = R \) of Householders makes it easy (and quite efficient!) to build a QR factorization.

- It turns out \( v' = a + \|a\|_2 \, e_1 \) works out, too—just pick whichever one causes less cancellation.

- \( H \) is symmetric

- \( H \) is orthogonal

**Demo:** 3x3 Householder demo [cleared]
Givens Rotations

If reflections work, can we make rotations work, too?

\[
\begin{bmatrix}
c & s \\
-s & c
\end{bmatrix}
\begin{bmatrix}
a_1' \\
a_2
\end{bmatrix} =
\begin{bmatrix}
\sqrt{a_1^2 + a_2^2} \\
0
\end{bmatrix}
\]

(produces 1 zero at a time)

**Demo:** 3x3 Givens demo [cleared]
Rank-Deficient Matrices and QR

What happens with QR for rank-deficient matrices?