Properties of ODEs

What is a linear ODE?

\[ f(t, \dot{x}) = A(t) \dot{x} + B(t) \]

What is a linear and homogeneous ODE?

\[ f(t, \dot{x}) = A(t) \dot{x} \]

What is a constant-coefficient ODE?

\[ f(t, x) = A \dot{x} + B \]
Properties of ODEs (II)

What is an autonomous ODE?

\[ f(t, x) = f(x) \]
\[ y'(t) = 1 \quad , \quad y_0(0) = 0 \]
Existence and Uniqueness

Consider the perturbed problem

\[ \begin{align*}
    y'(t) &= f(y) \\
    y(t_0) &= y_0
\end{align*} \]

Then if \( f \) is Lipschitz continuous (has ‘bounded slope’), i.e.

\[ \|f(y) - f(y)\| \leq L \|y - y\| \]

(where \( L \) is called the Lipschitz constant), then...

there exists a solution \( y \) in a neighborhood of \( t_0 \)

\[ \|y(t) - \hat{y}(t)\| \leq e^{L(t-t_0)} \|y - \hat{y}_0\| \]

What does this mean for uniqueness?

Implied by the bound. (\( \|y - \hat{y}_0\| = 0 \))
Unfortunate terminology accident: “Stability” in ODE-speak

To adapt to conventional terminology, we will use ‘Stability’ for

- the conditioning of the IVP, and
- the stability of the methods we cook up.

Some terminology:

An ODE is **stable** if and only if…

\[
\| y_0 - y_0 \| < \delta \implies \| y(t) - y(t) \| < \epsilon \quad \text{for all } t \geq t_0.
\]

An ODE is **asymptotically stable** if and only if

\[
\| y(t) - y(t) \| \to 0 \quad (t \to \infty)
\]

The solution is continuously dependent on initial data.
Example 1: Scalar, Constant-Coefficient

\[
\begin{align*}
\frac{y'}{y} &= \lambda \\
y'(t) &= \lambda y \\
y(0) &= y_0
\end{align*}
\]

where \( \lambda = a + ib \)

Solution?

\[
y(t) = y_0 \cdot e^{\lambda t} = y_0 e^{at} e^{ibt}
\]

When is this stable? \( a \leq 0 \)

\[ |\cos(bt) + i\sin(bt)| = 1 \]
Example II: Constant-Coefficient System

\[ A = VDV^{-1} \]

\[ \begin{align*}
\{ \ y'(t) &= Ay(t) \Rightarrow y'(t) = VDV^{-1}y(t) \\
y(t_0) &= y_0 \}
\end{align*} \]

Assume \( V^{-1}AV = D = \text{diag}(\lambda_1, \ldots, \lambda_n) \) diagonal. Find a solution.

\[ \begin{align*}
w'(t) &= Dw(t) \\
w(t_0) &= V^{-1}y_0
\end{align*} \]

When is this stable?

when \( \text{Re}(\lambda_i) \leq 0 \)
Euler’s Method

Discretize the IVP

\[
\begin{align*}
\begin{cases}
y'(t) &= f(y) \\
y(t_0) &= y_0
\end{cases}
\end{align*}
\]

- Discrete times: \( t_1, t_2, \ldots \), with \( t_{i+1} = t_i + h \)
- Discrete function values: \( y_k \approx y(t_k) \).

\[y(t) = y_0 + \int_{t_0}^{t} f(y(t)) \, dt\]

Rectangle Rule

\( \Rightarrow \) Euler’s method.
Euler’s method: Forward and Backward

\[ y(t) = y_0 + \int_{t_0}^{t} f(y(\tau)) \, d\tau, \]

Use ‘left rectangle rule’ on integral:

\[ y(t_k) = y_k \]

Forward Euler "explicit"

\[ y_{k+1} = y_k + h f(y_k) \quad \text{< need to eval expr.} \]
\[ h = \frac{t_{k+1} - t_k}{k} \quad \text{< assume constant} \]

Use ‘right rectangle rule’ on integral:

Back Euler "implicit"

\[ y_{k+1} = y_k + h f(y_{k+1}) \quad \text{< need to solve non-lin syst} \]

Demo: Forward Euler stability [cleared]
Global and Local Error

Let $u_k(t)$ be the function that solves the ODE with the initial condition $u_k(t_k) = y_k$. Define the local error at step $k$ as...

$$l_k = y_k - u_{k-1}(t_k)$$

Define the global error at step $k$ as...

$$g_k = y(t_n) - y_k$$
About Local and Global Error

Is global error = local errors?

No.

cf. "compound interest"

global error not accounted for by local "propagated error"

A time integrator is said to be accurate of order $p$ if…

$l_n = O(h^{p+1})$
A time integrator is said to be accurate of order $p$ if $\ell_k = O(h^{p+1})$

This requirement is one order higher than one might expect—why?

\[
\integrate \text{ to } t=1, \text{ # steps } \frac{1}{h} \\
\frac{1}{h} \cdot O(h^{p+1}) = O(h^p) \\
e^{rt} \approx 1 + rt
\]
Stability of a Method

Find out when forward Euler is stable when applied to $y'(t) = \lambda y(t)$.

$$y_k = y_{k-1} + h \cdot \lambda y_{k-1}$$

$$= (1 + h\lambda)^k \cdot y_0$$

$|1 + h\lambda| \leq 1 \Rightarrow$ stable. $\|y_k\| \leq |1 + h\lambda|^k \cdot \|y_0\|$

\[\text{amplification factor} = \text{Im}(\lambda) \]
\[\text{Re}(\lambda)\]
Stability: Systems

What about stability for systems, i.e.

\[ y'(t) = Ay(t) ? \]

\[ w = V^{-1} y \]

\[ \Rightarrow |1 + h\lambda i| \leq 1 \text{ implies stability.} \]