# Solving a Linear System

. FP

- norm (vector)

Given:

 $\triangleright m \times n \text{ matrix } A$ 

- conditioning  $\rightarrow$  m-vector **b** 

What are we looking for here, and when are we allowed to ask the question?

$$A = \int_{-\infty}^{\infty} - Solution may not exist - Solution may not be unique. 
 $m = n$ ,  $A^{-1}$  exists.$$

Next: Want to talk about conditioning of this operation. Need to measure distances of matrices.

#### Matrix Norms

What norms would we apply to matrices?

#### Intuition for Matrix Norms

Provide some intuition for the matrix norm.

$$y = \frac{x}{||x||}$$

$$max \frac{||Ax||}{||x||} = \frac{||A||}{||x||} \cdot ||x||| = max \frac{||Ay||}{||x||} \cdot ||x|||$$

$$||x|| = ||x|| + ||x||$$

### Matrix Norm Properties

What is  $||A||_{\underline{1}}$ ?  $||A||_{\underline{\infty}}$ ?

$$||All_i| = \max_{||x||=1} \frac{||Ax||_1}{||x||_1} = \max_{j} \frac{||Aij||}{||Aij||}, ||A||_{\infty}$$

How do matrix and vector norms relate for  $n \times 1$  matrices?

Demo: Matrix norms [cleared]

# Properties of Matrix Norms



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Matrix norms inherit the vector norm properties:

- $||A|| > 0 \Leftrightarrow A \neq 0.$
- $\blacktriangleright \ \|\gamma \mathbf{A}\| = |\gamma| \, \|\mathbf{A}\| \text{ for all scalars } \gamma.$
- ▶ Obeys triangle inequality  $||A + B|| \le ||A|| + ||B||$

But also some more properties that stem from our definition:



### Conditioning

What is the condition number of solving a linear system Ax = b?

In pals: b with error 
$$\Delta b$$

Output:  $\times$  with error  $\Delta s$ 

rel. Gar. in out  $\frac{||\Delta x|| / ||x|||}{||\Delta x|| / ||x|||} = \frac{||\Delta x|| / ||x|||}{||\Delta x|| / ||x|||}$ 
 $A \times b = A \times b =$ 

## Conditioning of Linear Systems: Observations

Showed  $\kappa(\text{Solve } A\mathbf{x} = \mathbf{b}) \leq \|A^{-1}\| \|A\|^{\epsilon}$ . The found an upper bound on the condition number. With a little bit of fiddling, it's not too hard to find examples that achieve this bound, i.e. that it is *sharp*.

So we've found the condition number of linear system solving, also called the condition number of the matrix A:

$$\operatorname{cond}(A) = \kappa(A) = \|A\| \|A^{-1}\|.$$

$$\operatorname{cond}_{\mathcal{L}}(A) = \|A\|_{\mathcal{L}} \|A^{-1}\|_{\mathcal{L}}$$

$$\operatorname{cond}(\mathcal{L}) = \|A\| \|A^{-1}\| \le \|A\| \|A^{-1}\| = \operatorname{cond}(A)$$

## Conditioning of Linear Systems: More properties

cond is relative to a given norm. So, to be precise, use

$$\mathsf{cond}_2$$
 or  $\mathsf{cond}_\infty$ .

Ax-6 solve c

If  $A^{-1}$  does not exist:  $\operatorname{cond}(A) = \infty$  by convention. A year way What is  $\kappa(A^{-1})$ ?

What is the condition number of matrix-vector multiplication?

**Demo:** Condition number visualized [cleared] **Demo:** Conditioning of 2x2 Matrices [cleared]

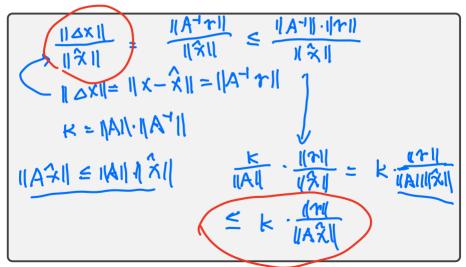
#### Residual Vector

What is the residual vector of solving the linear system

$$\mathbf{b} = A\mathbf{x}$$
?

### Residual and Error: Relationship

How do the (norms of the) residual vector  $\mathbf{r}$  and the error  $\Delta \mathbf{x} = \mathbf{x} - \hat{\mathbf{x}}$  relate to one another?



## Changing the Matrix

So far, all our discussion was based on changing the right-hand side, i.e.

$$A\mathbf{x} = \mathbf{b} \quad \rightarrow \quad A\widehat{\mathbf{x}} = \widehat{\mathbf{b}}.$$

The matrix consists of FP numbers, too-it, too, is approximate. I.e.

$$A\mathbf{x} = \mathbf{b} \quad \rightarrow \quad \hat{A}\hat{\mathbf{x}} = \mathbf{b}.$$

What can we say about the error now?

## Changing Condition Numbers

Once we have a matrix $A$ in a linear system $A\mathbf{x} = \mathbf{b}$ , are we stu condition number? Or could we improve it?	ck with its
What is this called as a general concept?	

# In-Class Activity: Matrix Norms and Conditioning

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