### Changing the Matrix

So far, only discussed changing the RHS, i.e.  $A\mathbf{x} = \mathbf{b} \rightarrow A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ . The matrix consists of FP numbers, too—it, too, is approximate. I.e.

$$A\mathbf{x} = \mathbf{b} \quad \rightarrow \quad \widehat{A}\widehat{\mathbf{x}} = \mathbf{b}.$$

What can we say about the error due to an approximate matrix?

$$\hat{\lambda} = \chi + \Delta \chi \qquad \hat{A} = A + \Delta A$$

$$\frac{\|\Delta \chi\|}{\|\hat{\chi}\|} ? \qquad = A^{-1}(A\hat{\chi} - \hat{A}\hat{\chi}) = -A^{-1}\Delta A\hat{\chi}$$

$$= K \cdot \|\Delta \chi\| \leq \|A^{-1}\| \cdot \|\Delta A\| \cdot \|\hat{\chi}\|$$

$$= K \cdot \|A\| \qquad ||\Delta \chi|| \leq \|A^{-1}\| \cdot \|\Delta A\| \cdot \|\hat{\chi}\|$$

$$= K \cdot \|A\| \qquad ||\Delta \chi|| \leq \|A^{-1}\| \cdot \|\Delta A\| \cdot \|\hat{\chi}\|$$

$$= K \cdot \|A\| \qquad ||\Delta \chi|| \leq \|A^{-1}\| \cdot \|\Delta A\| \cdot \|\hat{\chi}\|$$

$$= K \cdot \|A\| \cdot \|A\| \cdot \|A\| \cdot \|A\| \cdot \|A\| \cdot \|A\|$$

### Changing Condition Numbers

Once we have a matrix A in a linear system  $A\mathbf{x} = \mathbf{b}$ , are we stuck with its condition number? Or could we improve it?

$$\frac{DAx = Db}{ADx} = \frac{1}{1} \frac$$

What is this called as a general concept?

- Left preconditioning 
$$\pi = (DA)^T(Pb)$$
  
- Right precon.  $\bar{\chi} = (AD)^Tb$ ,  $\chi = D\bar{\chi}$ 

# In-Class Activity: Matrix Norms and Conditioning



In-class activity: Matrix Norms and Conditioning

# Singular Value Decomposition (SVD)

What is the Singular Value Decomposition of an  $m \times n$  matrix?

$$A = U \sum V^{T}$$
orth  $\sum V^{T} V^{T}$ 

# Computing the 2-Norm

Using the SVD of *A*, identify the 2-norm.

$$A = U \Sigma V^T$$
 $- ||A||_2 = ||Z||_2 = ||max|$ 
 $- ||QB||_2 = ||B||_2 = ||BQ||_2, Q oth$ 

Express the matrix condition number  $cond_2(A)$  in terms of the SVD:

#### Not a matrix norm: Frobenius

The 2-norm is very costly to compute. Can we make something simpler?

What about its properties?

- 
$$||A||_{F} > 0$$
,  $||A||_{F} = 0 \Leftrightarrow A = 0$   
-  $||A||_{F} = \lambda ||A||_{F}$   
-  $||A + B||_{F} \leq ||A||_{F} + 1||B||_{F}$ 

### Frobenius Norm: Properties

Is the Frobenius norm induced by any vector norm?

How does it relate to the SVD?

$$||A||_F = \int_{\mathcal{F}} \mathcal{G}_{\mathcal{F}}^2$$

$$||A||_F = \int_{\mathcal{F}} \mathcal{G}_{\mathcal{F}}^2$$
\*Errata: the square root was missing from the original scribbles

### Solving Systems: Simple cases



Solve Dx = b if D is diagonal. (Computational cost?)

Solve Qx = b if Q is orthogonal. (Computational cost?)

Given SVD  $A = U\Sigma V^T$ , solve  $A\mathbf{x} = \mathbf{b}$ . (Computational cost?)

# Solving Systems: Triangular matrices

Solve

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

#### Demo: Coding back-substitution [cleared]

What about non-triangular matrices?



#### Gaussian Elimination

#### Demo: Vanilla Gaussian Elimination [cleared]

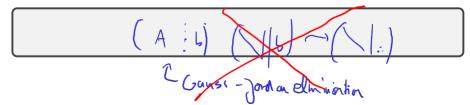
What do we get by doing Gaussian Elimination?



How is that different from being upper triangular?



What if we do not just eliminate downward but also upward?



#### LU Factorization

#### What is the LU factorization?



### Solving Ax = b

Does LU help solve 
$$Ax = b$$
?

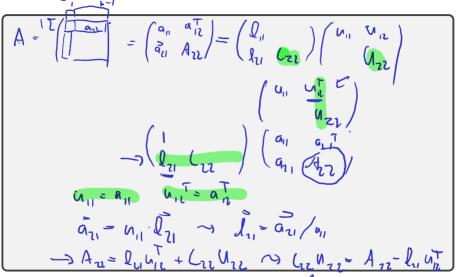
$$A = b$$

$$C(x) = b$$

$$C(y) = b = f_m \text{ subst.} \Rightarrow O(n^2)$$

$$(x) = y = b_m \text{ subst.} \Rightarrow O(n^2)$$

# Determining an LU factorization



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Demo: LU Factorization [cleared]

### Computational Cost

What is the computational cost of multiplying two  $n \times n$  matrices?

$$(AB)_{ij} = \sum_{k} Q_{ik} b_{kj} \qquad 0 (n^{3})$$

$$0 (n^{k}) u_{11} = a_{11}, u_{12}^{T} = a_{12}^{T}.$$

$$0 (n^{k}) I_{21} = a_{21}/u_{11}.$$

$$0 (n^{k}) L_{22}U_{22} = A_{22} - I_{21}u_{12}^{T}.$$

What is the computational cost of carrying out LU factorization on an  $n \times n$  matrix?

Demo: Complexity of Mat-Mat multiplication and LU [cleared]