Changing the Matrix

So far, only discussed changing the RHS, i.e. $A\boldsymbol{x} = \boldsymbol{b} \quad \to \quad A\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{b}}$.
The matrix consists of FP numbers, too—it, too, is approximate. I.e.

$$\underline{A\boldsymbol{x} = \boldsymbol{b}} \quad \to \quad \boxed{\widehat{A}\widehat{\boldsymbol{x}} = \boldsymbol{b}.}$$

What can we say about the error due to an approximate matrix?

$\hat{x} = x + \Delta x \qquad \hat{A} = A + \Delta A$

$\dfrac{\|\Delta x\|}{\|\hat{x}\|}$?

$\Delta x = \hat{x} - x = \hat{x} - A^{-1}b = A^{-1}(A\hat{x} - b)$

$= A^{-1}(A\hat{x} - \hat{A}\hat{x}) = \underline{-A^{-1}\Delta A\hat{x}}$

$\|\Delta x\| \leq \|A^{-1}\| \cdot \|\Delta A\| \cdot \|\hat{x}\|$

$= \|-A^{-1}\Delta A\hat{x}\|$

$\leq \kappa \cdot \dfrac{\|\Delta A\|}{\|A\|}$

$\kappa = \|A\| \cdot \|A^{-1}\|$

Changing Condition Numbers

Once we have a matrix A in a linear system $A\boldsymbol{x} = \boldsymbol{b}$, are we stuck with its condition number? Or could we improve it?

$\underline{DAx = Db}$. $\|DA\| \cdot \|A^{-1}D^{-1}\| \geq \|I\| = 1$

$\underline{AD\bar{x} = b}$ $\|AD\| \cdot \|D^{-1}A^{-1}\|$.

What is this called as a general concept?

- Left preconditioning $x = (DA)^{-1}(Db)$
- Right precon. $\bar{x} = (AD)^{-1}b$, $x = D\bar{x}$

In-Class Activity: Matrix Norms and Conditioning

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In-class activity: Matrix Norms and Conditioning

Singular Value Decomposition (SVD)

What is the *Singular Value Decomposition* of an $m \times n$ matrix?

$$A = \underbrace{U}_{\text{orth}} \underbrace{\Sigma}_{\text{Diag}} \underbrace{V^T}_{\text{orth.}}$$

$$\begin{cases} U^T U = I \\ V^T V = I \end{cases}$$

- col. of U: left sing. vec.
- col. of V: righ sing vec.
- sing. values: $\sigma_1 \dots \sigma_m$

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{pmatrix}$$

Computing the 2-Norm

Using the SVD of A, identify the 2-norm.

$A = U\Sigma V^T$

- $\|A\|_2 = \|\Sigma\|_2 = \sigma_{max}$
- $\|QB\|_2 = \|B\|_2 = \|BQ\|_2$, Q orth.

Express the matrix condition number $\text{cond}_2(A)$ in terms of the SVD:

$\text{cond}_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = \sigma_{max} / \sigma_{min}$

Not a matrix norm: Frobenius

The 2-norm is very costly to compute. Can we make something simpler?

$$\|A\|_F = \sqrt{\sum_i \sum_j |a_{ij}|^2}$$

Frobenius norm.

What about its properties?

- $\|A\|_F \geq 0$, $\|A\|_F = 0 \Leftrightarrow A = 0$
- $\|\lambda A\|_F = \lambda \|A\|_F$
- $\|A+B\|_F \leq \|A\|_F + \|B\|_F$.

Frobenius Norm: Properties

Is the Frobenius norm induced by any vector norm?

$$\|A\| = \max_{\|x\|=1} \|Ax\| \qquad \|I\|_F = \sqrt{n}$$

How does it relate to the SVD?

$$\|A\|_F = \sqrt{\sum_i \sigma_i^2}$$

$$\|A\|_F = \sqrt{\mathrm{tr}(AA^T)}$$

* Errata: the square root was missing from the original scribbles

Solving Systems: Simple cases

Solve $D\boldsymbol{x} = \boldsymbol{b}$ if D is diagonal. (Computational cost?)

$x_i = b_i / D_{ii}$ $\qquad O(n)$

Solve $Q\boldsymbol{x} = \boldsymbol{b}$ if Q is orthogonal. (Computational cost?)

$x = Q^T b$ $\qquad O(n^2)$

Given SVD $A = U\Sigma V^T$, solve $A\boldsymbol{x} = \boldsymbol{b}$. (Computational cost?)

$U \Sigma V^T x = b$

$\Sigma \underbrace{V^T x}_{y} = U^T b$

$y = V^T x$

$V y = x$

$O(n^2) \rightarrow O(n^3)$ to find the SVD

Solving Systems: Triangular matrices

Solve

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

$a_{33} z + a_{34} w = b_3$

$a_{44} w = b_4$

Demo: Coding back-substitution [cleared]

What about non-triangular matrices?

Gaussian el.

Gaussian Elimination

Demo: Vanilla Gaussian Elimination [cleared]

What do we get by doing Gaussian Elimination?

REF

How is that different from being upper triangular?

→ LU

What if we do not just eliminate downward but also upward?

$(A \,\vdots\, b)$

Gauss-Jordan elimination

What is the LU factorization?

$A = LU$

- L lower, diag of all ones
- U upper

$a_{11} = l_{11} \cdot u_{11}$

Solving $A\boldsymbol{x} = \boldsymbol{b}$

Does LU help solve $A\boldsymbol{x} = \boldsymbol{b}$?

$A = LU$

$Ax = b$

$L\underbrace{Ux}_{y} = b$

$Ly = b$ ← fw subst → $O(n^2)$

$Ux = y$ ← bw subst. → $O(n^2)$

Determining an LU factorization

$$A = \begin{pmatrix} a_{11} & a_{12}^T \\ \vec{a}_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} l_{11} & \\ l_{21} & L_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ & U_{22} \end{pmatrix}$$

$$\begin{pmatrix} u_{11} & u_{12}^T \\ & U_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ l_{21} & L_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}^T \\ a_{21} & A_{22} \end{pmatrix}$$

$u_{11} = a_{11}$ $\quad$ $u_{12}^T = a_{12}^T$

$\vec{a}_{21} = u_{11} \cdot \vec{l}_{21} \rightsquigarrow \vec{l}_{21} = \vec{a}_{21} / u_{11}$

$\rightarrow A_{22} = l_{21} u_{12}^T + L_{22} U_{22} \rightsquigarrow L_{22} U_{22} = A_{22} - l_{21} u_{12}^T$

$(n-1) \times (n-1)$

Demo: LU Factorization [cleared]

Computational Cost

What is the computational cost of multiplying two $n \times n$ matrices?

$(AB)_{ij} = \sum_k a_{ik} b_{kj}$ $\quad O(n^3)$

- $O(n)$ $u_{11} = a_{11}$, $\boldsymbol{u}_{12}^T = \boldsymbol{a}_{12}^T$.
- $O(n)$ $\boldsymbol{l}_{21} = \boldsymbol{a}_{21}/u_{11}$.
- $O(n^2)$ $L_{22}U_{22} = A_{22} - \boldsymbol{l}_{21}\boldsymbol{u}_{12}^T$.

$O(n^2) + O(n-1)^2 + \dots$

What is the computational cost of carrying out LU factorization on an $n \times n$ matrix?

$O(n^3)$

Demo: Complexity of Mat-Mat multiplication and LU [cleared]