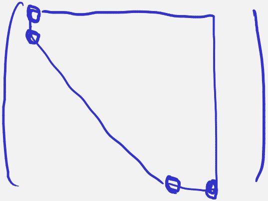
QR Iteration: Computational Expense

A full QR factorization at each iteration costs $O(n^3)$–can we make that cheaper?

Upper Hessenberg form:

$A = Q$  Q^T

↳ carrying out QR fact (in an it. of QR iteration) for X_k in UHess form costs $O(n^2)$ / QR instead of $O(n^3)$

Demo: Householder Similarity Transforms [cleared]

QR/Hessenberg: Overall procedure

Overall procedure:

1. Reduce matrix to Hessenberg form $\to$ using Householders
2. Apply QR iteration using Givens QR to obtain Schur form

For symmetric matrices:

- Use Householders to attain tridiagonal form
- Use QR iteration with Givens to attain diagonal form

Krylov space methods: Intro

What subspaces can we use to look for eigenvectors?

QR: $\text{span}\{A^\ell y_1, A^\ell y_2, \dots, A^\ell y_k\}$

Krylov space: $\text{span}\{\underbrace{x}_{x_0}, \underbrace{Ax}_{x_1}, \dots, \underbrace{A^{k-1}x}_{x_{k-1}}\}$

$$K_k := [x_0 \cdots x_{k-1}]$$

Krylov for Matrix Factorization

What matrix factorization is obtained through Krylov space methods?

$$A K_n = [x_1, x_2, \cdots x_n]$$

$$= K_n \underbrace{[e_2, e_3 \cdots, e_n, K_n^{-1} x_n]}_{C_n}$$

$$\begin{bmatrix} 0 & & & \vdots \\ 1 & 0 & & \vdots \\ & 1 & \ddots & \vdots \\ & & \ddots & \vdots \end{bmatrix}$$

- $C_n = K_n^{-1} A K_n$ upper Hessenberg
- ∴ by construction "upper Hessenberg."
- works well with just matvec

Conditioning in Krylov Space Methods/Arnoldi Iteration (I)

What is a problem with Krylov space methods? How can we fix it?

K_k can be ill-cond. $A^{n-2}x$, $A^{n-1}x$

- orthogonalize.

$$Q_n R_n = K_n \Rightarrow Q_n = K_n R_n^{-1}$$

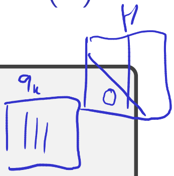
$$\Rightarrow Q_n^T A Q_n = R_n \underbrace{K_n^{-1} A K_n}_{C_n} R_n^{-1}$$

- C_n is upper-Hess.
- $Q_n^T A Q_n$ " "

Conditioning in Krylov Space Methods/Arnoldi Iteration (II)

$$Q_n^T A Q_n = H \iff AQ_n = Q_n H$$

kth col



(*) $Aq_k = h_{1k} q_1 + \dots + h_{k+1,k} q_{k+1}$

Multiply by q_j^T from the left to find

$$h_{jk} = q_j^T A q_k$$

For symmetric: Lanczos

Demo: Arnoldi Iteration [cleared] (Part 1)

Krylov: What about eigenvalues?

How can we use Arnoldi/Lanczos to compute eigenvalues?

$$Q_k = \left[\, Q_k^{\text{have}} \;\middle|\; \underbrace{Q_k^{\neg\text{have}}}_{0} \,\right]$$

$$H = Q^T A Q = \begin{bmatrix} \text{have} \\ \neg\text{have} \end{bmatrix} A \begin{bmatrix} \text{have} & \neg\text{have} \end{bmatrix} =$$

Ritz values: eigenvalues of

Demo: Arnoldi Iteration [cleared] (Part 2)

Computing the SVD (Kiddy Version)

How can I compute an SVD of a matrix A?