About Convergence Rates

Demo: Rates of Convergence [cleared]

Characterize linear, quadratic convergence in terms of the 'number of accurate digits'.

- Linear.

  $$\|e_{k+1}\| \le C \cdot \|e_k\|, \quad C \in (0,1)$$

  each iter gains constant number of digits

- Quadratic.

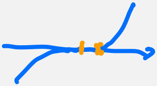
  $$\|e_{k+1}\| \le C \cdot \|e_k\|^2$$

Stopping Criteria

Comment on the 'foolproof-ness' of these stopping criteria:

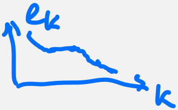
1. $|f(x)| < \varepsilon$ ('residual is small')
2. $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| < \varepsilon$
3. $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| \,/\, \|\boldsymbol{x}_k\| < \varepsilon$

1.



2. not relative

3.



No silver bullet!

Bisection Method

Demo: Bisection Method [cleared]

What's the rate of convergence? What's the constant?

Linear, $1/2$

Fixed Point Iteration

$f(x)=0$

$x^2-x-2=0$

$$\begin{aligned} x_0 &= \langle\text{starting guess}\rangle \\ x_{k+1} &= g(x_k) \end{aligned}$$

$\underbrace{x^2-2}_{g(x)}=x$

Demo: Fixed point iteration [cleared]

When does fixed point iteration converge? Assume g is smooth.

Let x^* be the fixed point $x^*=g(x^*)$

If $|g'(x^*)|<1$ at the FP, then there's a nbh with conv.

$e_{k+1}=x_{k+1}-x^*=g(x_k)-g(x^*)$

Fixed Point Iteration: Convergence cont'd.

Error in FPI: $e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$ $\longleftarrow$

$$g(x_k) - g(x^*) = g'(\theta_k) \cdot (x_k - x^*) = g'(\theta_k) \cdot e_k$$

$$\theta_k \in (x_k, x^*)$$

$e_{k+1} = g'(\theta_k) \cdot e_k \Rightarrow$ linear convergence with constant from upper bound of g'

What if $g'(x^*) = 0$?

$$g(x_k) - g(x^*) = g''(\xi_k) \cdot \frac{(x_k - x^*)^2}{2}$$

$$= g''(\xi_k) \frac{e_k^2}{2}$$

Newton's Method

Derive Newton's method.

$$f(x_k) + f'(x_k) \cdot h = 0 \iff h = -\frac{f(x_k)}{f'(x_k)}$$

$$x_0 = (\text{starting guess})$$

$$x_{k+1} = x_k + h = x_k - \frac{f(x_k)}{f'(x_k)}$$

Convergence and Properties of Newton

What's the rate of convergence of Newton's method?

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = g(x_k)$$

$x = x^*$

$$g'(x) = 1 - \frac{f'(x)\cdot f'(x) - f(x) f''(x)}{(f'(x))^2} = \frac{\overset{0}{f(x)} f''(x)}{(f'(x))^2}$$

multiplicity: $=0$

$\neq 0 \Rightarrow$ quadratic

Drawbacks of Newton?

- only good with good initial guess.
- need $f'(x)$.

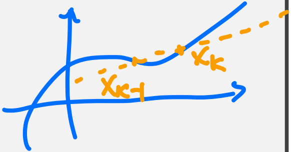
Demo: Newton's method [cleared]

Demo: Convergence of Newton's Method [cleared]

Secant Method

What would Newton without the use of the derivative look like?

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$



$$x_{k+1} = x_k - \frac{f(x_k)}{\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}}$$

Convergence of Properties of Secant

Rate of convergence (not shown) is $\left(1 + \sqrt{5}\right)/2 \approx 1.618$.

Drawbacks of Secant?

- local.
- 2 inits needed.

Demo: Secant Method [cleared]
Demo: Convergence of the Secant Method [cleared]

Secant (and similar methods) are called Quasi-Newton Methods.

Root Finding with Interpolants

Secant method uses a linear interpolant based on points $f(x_k)$, $f(x_{k-1})$, could use more points and higher-order interpolant:

What about existence of roots in that case?

Achieving Global Convergence

The linear approximations in Newton and Secant are only good locally. How could we use that?

In-Class Activity: Nonlinear Equations

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