Gaussian Quadrature

So far: nodes chosen from outside.
Can we gain something if we let the quadrature rule choose the nodes, too? Hope: More design freedom → Exact to higher degree.

n nodes
n weights

Σ $2n$ things to mess with

$\Rightarrow$ node $2n$ wishes
hope to integrate $2n$ polys

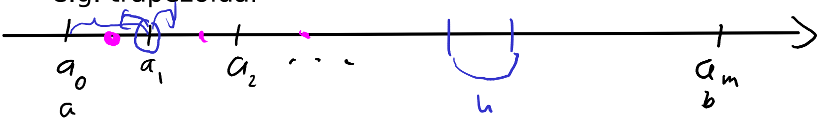
Demo: Gaussian quadrature weight finder [cleared]

Composite Quadrature

High-order polynomial interpolation requires a high degree of smoothness of the function.

Idea: Stitch together multiple lower-order quadrature rules to alleviate smoothness requirement.

e.g. trapezoidal



Error in Composite Quadrature

What can we say about the error in the case of composite quadrature?

Error for one panel $\left|\int f - p_{n-1}\right| \leq C \cdot \|f^{(n)}\|_\infty \cdot h^{n+1}$

$$\left|\int_a^b f(x)\,dx - \underbrace{\sum_{j=1}^{m}}_{\text{sum over intervals}} \overbrace{\sum_{i=1}^{n}}^{\text{sum over nodes}} w_j f(x_{i,j})\right|$$

$$\leq C\|f^{(n)}\|_\infty \cdot \sum_{j=1}^{m} (a_j - a_{j-1})^{n+1}$$

$$\leq C \cdot \|f^{(n)}\|_\infty \cdot \sum_{j=1}^{m} \underbrace{(a_j - a_{j-1})^n}_{\leq h^n} (a_j - a_{j-1})$$

$$\leq C \cdot \|f^{(n)}\|_\infty (b-a)\, h^n$$

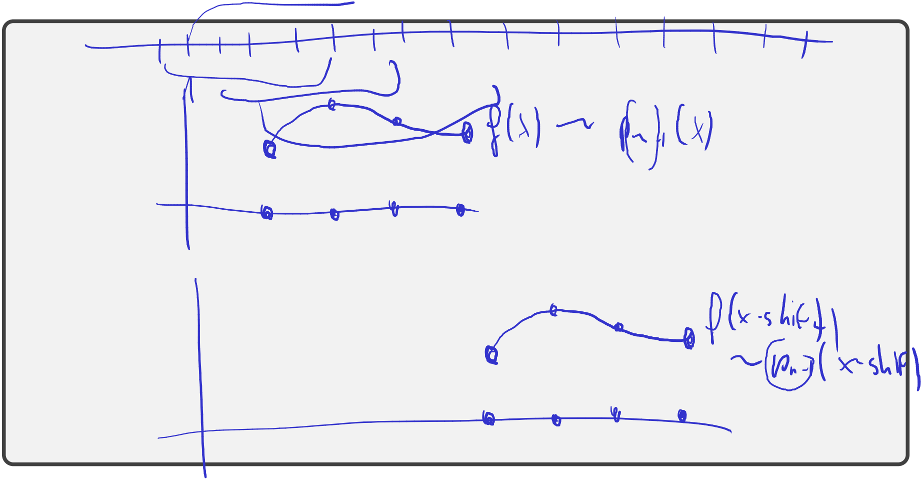
Composite Quadrature: Notes

Observation: Composite quadrature loses an order compared to non-composite.

Idea: If we can estimate errors on each subinterval, we can shrink (e.g. by splitting in half) only those contributing the most to the error.
(adaptivity, $\rightarrow$ hw)

Taking Derivatives Numerically

Why *shouldn't* you take derivatives numerically?



Demo: Taking Derivatives with Vandermonde Matrices [cleared]

Taking Derivatives Numerically

Why *shouldn't* you take derivatives numerically?

- "unbounded"

  $\|[\sin(kx)]'\|_\infty = k\|\cos(kx)\|_\infty = k$

  $\|\sin(kx)\|_\infty = 1$

- "noise amplification" $\hat{f} = f + e$ $\quad \|\hat{f}'\| \geq \underline{\|e'\|} - \underline{\|f'\|}$

- cancellation error $\quad f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

- Less accurate.

  interp : $\sim h^n$ $\qquad$ diff $\sim h^{n-1}$

  quad : $\sim h^{n+1}$

Demo: Taking Derivatives with Vandermonde Matrices [cleared]

Finite Differences

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

forward difference.

first order.

$$f(x+h) = f(x) + h \cdot f'(x) + O(h^2)$$

$$\Rightarrow \text{err} = O(h)$$

More Finite Difference Rules

Similarly:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

(Centered differences)

Can also take higher order derivatives:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

Can find these by trying to match Taylor terms.
Alternative: Use linear algebra with interpolate-then-differentiate to find FD formulas.
Demo: Finite Differences vs Noise [cleared]
Demo: Floating point vs Finite Differences [cleared]