Stability of a Method

$\rightarrow \; y'(t) = f(y(t)) \qquad y(0) = y_0$

Find out when forward Euler is stable when applied to $y'(t) = \lambda y(t)$.

$y' = \lambda y$

$\mathrm{Re}\,\lambda \le 0$ stable

$\mathrm{Re}\,\lambda < 0$ asympt. stable

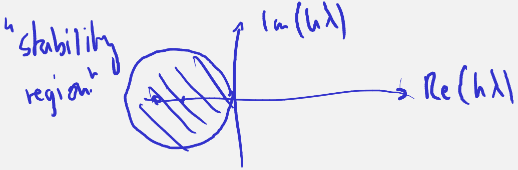
$$y_{k+1} = y_k + h f(y_k)$$

$$= y_k(1 + h\lambda)$$

$$= y_0 \underbrace{(1 + h\lambda)^{k+1}}_{|\cdot| \le 1}$$

$$\|y_{k+1}\| \le \|y_k\|$$

$$\Leftrightarrow \quad |1 + h\lambda| \le 1$$



Stability: Systems

What about stability for systems, i.e.

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t)?$$

$$D = VAV^{-1} \iff V^{-1}DV = A$$

$$w = Vy$$

$$\iff V^{-1}w = y$$

$$y' = Ay$$

$$V^{-1}w' = V^{-1}DVV^{-1}w$$

$$w' = Dw$$

Stability: Nonlinear ODEs

What about stability for nonlinear systems, i.e.

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t))?$$

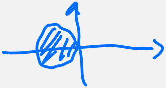
Consider perturbation $e(t) = y(t) - \hat{y}(t)$

$$e'(t) = f(y(t)) - f(\hat{y}(t)) \approx J_f(y(t))\, e(t)$$

Stability for Backward Euler

Find out when backward Euler is stable when applied to $y'(t) = \lambda y(t)$.

F.E.



$h\lambda$

B.E.

$$y_k = y_{k-1} + h\cdot\lambda\cdot y_k$$

$$\Rightarrow \quad y_k = \frac{1}{1-h\lambda}\, y_{k-1} \cdots = \left(\frac{1}{1-h\lambda}\right)^k y_0$$

stable if $|1-h\lambda| \geq 1$

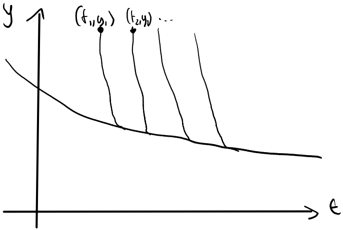
if $\mathrm{Re}(\lambda) \leq 0$, stable for all h. "unconditionally stable".

Demo: Backward Euler stability [cleared]

Stiff ODEs: Demo

Demo: Stiffness [cleared]

'Stiff' ODEs



- Stiff problems have *multiple time scales.*
  (In the example above: Fast decay, slow evolution.)
- In the case of a stable ODE system

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t)),$$

stiffness can arise if $J_{\boldsymbol{f}}$ has eigenvalues of very different magnitude.

Stiffness: Observations

Why not just 'small' or 'large' magnitude?

"multi-timescale".

What is the problem with applying explicit methods to stiff problems?

inefficient (small h needed)

Stiffness vs. Methods

Phrase this as a conflict between accuracy and stability.

- Accuracy (capturing the slow time scale) can be achieved with large h.
- Explicit methods demands small h for stability.

Can an implicit method take arbitrarily large time steps?

- stability ✓
- accuracy ✗.

Predictor-Corrector Methods

Idea: Obtain intermediate result, improve it (with same or different method).

Predict with FE $\quad \hat{y}_{k+1} = y_k + h f(y_k)$

Correct with trapezoidal rule: $\quad y_{k+1} = y_k + h\left(\frac{f(y_k) + f(\hat{y}_{k+1})}{2}\right)$

Heun's method

Runge-Kutta/'Single-step'/'Multi-Stage' Methods

Idea: Compute intermediate 'stage values', compute new state from those:

$$\to r_1 = f(t_k + c_1 h,\ y_k + h(a_{11} r_1 + \cdots + a_{1s} r_s))$$

$$r_2 = \quad c_2 \quad a_{21} \quad a_{2s}$$

$$\vdots$$

$$r_s \quad c_s \quad a_{s1} \quad a_{ss}$$

$$y_{k+1} = y_k + h(b_1 r_1 + \cdots + b_s r_s)$$

Can summarize in a *Butcher tableau*:

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

$y^1 = 1$

$\to c_i = \sum a_{ji}$

Runge-Kutta: Properties

When is an RK method explicit?

0 above and on the diagonal of the a matrix

When is it implicit?

if that's not the case

When is it *diagonally implicit*? (And what does that mean?)

If everything in the a matrix above
the diag is zero