\[ q_i = v_i - \left( \frac{q_i^T v_i}{\|q_i\|^2} \right) \cdot q_i, \]

\[ q_i^T q_i = q_i^T v_i - \frac{q_i^T v_i}{\|q_i\|^2} \cdot q_i^T q_i \]
Schur form

Show: Every matrix is orthonormally similar to an upper triangular matrix, i.e. \( A = QUQ^T \). This is called the Schur form or Schur factorization.

\[
A v = \lambda v \\
V = \text{Span}(v) \\
Q_i = \left\{ u : u^T v = 0 \right\} \\
V = \left\{ w : w^T v = 0 \right\}
\]

\[
A = Q_i \begin{pmatrix} \lambda & 2 \\ 0 & 0 \end{pmatrix} Q_i^T
\]

Schur form: \( A = Q_i \begin{pmatrix} \lambda & 2 \\ 0 & 0 \end{pmatrix} Q_i^T \) unique? Maybe not.
Schur Form: Comments, Eigenvalues, Eigenvectors

$A = QUQ^T$. For complex $\lambda$:

- Either complex matrices, or
- $2 \times 2$ blocks on diag.

If we had a Schur form of $A$, how can we find the eigenvalues?

And the eigenvectors?

\[
U - \lambda \mathbf{I} = \begin{pmatrix}
\begin{matrix}
(-\lambda \mathbf{I}) & \mathbf{u}^* \\
\mathbf{u} & \mathbf{0}
\end{matrix}
\end{pmatrix}
\]

\[
x = \begin{bmatrix}
\mathbf{u}^* \\
-1
\end{bmatrix}
\]

$(U - \lambda \mathbf{I}) x = \mathbf{0}$

$U \mathbf{x} = \lambda \mathbf{x}$

$y = Qx \Rightarrow Ay = \lambda y$
Computing Multiple Eigenvalues

All Power Iteration Methods compute one eigenvalue at a time. What if I want all eigenvalues?

1. "Deflation": Similarity transform to \((A, B)\), find eigenvec of B keep going

2. Simultaneous Iteration
Simultaneous Iteration

What happens if we carry out power iteration on multiple vectors simultaneously?

$x_0 = \text{some random matrix}$

$x_{k+1} = A \cdot x_k$

Problems:

- need to normalize
- $x_i$ gets ill-conditioned, all columns converge to dominant eigenvector
Orthogonal Iteration

\[ x_0 \in \mathbb{R}^{n \times p} \ (p \leq n) \]

\[ \text{for } k = 0, 1, \ldots \]

\[ Q_k R_k = X_k \]

\[ X_{k+1} = A Q_k \]

\[ \bullet \text{ converges.} \]

\[ \Theta \text{ linear conv. expensive} \]
Toward the QR Algorithm

\[
\begin{align*}
Q_0 R_0 &= x_0 \\
X_1 &= AQ_0 \\
Q_1 R_1 &= x_1 = AQ_0 \implies Q_1 R_1 Q_0^T &= A
\end{align*}
\]

Once \( Q_{n+1} \approx Q_n \), \( Q^* R^* Q^* \) is desired criterion.

**Demo:** Orthogonal Iteration [cleared]
**QR Iteration/QR Algorithm**

**Ortho. iter.**

\[ x_0 = A \]

\[ Q_K R_K = X_K \]

\[ X_{K+1} = A Q_K. \]

\[ Q_K = \overline{Q}_0 \overline{Q}_1 \cdots \overline{Q}_K \]

**QR iter.**

\[ \overline{x}_0 = A \]

\[ \overline{Q}_K \overline{R}_K = \overline{X}_K \]

\[ \overline{X}_{K+1} = \overline{R}_K \overline{Q}_K. \]

\[ Q_K = \overline{Q}_0 \overline{Q}_1 \cdots \overline{Q}_K \]

**Shift form**
QR Iteration: Incorporating a Shift

How can we accelerate convergence of QR iteration using shifts?

\[
\begin{align*}
\bar{X}_0 &= A \\
\bar{Q}_k &\bar{Q}_k = \bar{X}_k - \sigma_k I \\
\bar{X}_{k+1} &= \bar{Q}_k \bar{Q}_k + \sigma_k I \\
\bar{X}_{k+1} &= \bar{Q}_k \bar{X}_k \bar{Q}_k + \sigma_k I \\
\bar{X}_{k+1} &= \bar{Q}_k \bar{X}_k \bar{Q}_k
\end{align*}
\]

**Shift:**
- Bottom right only of \( \bar{X}_k \)
- Bottom right 2×2, find eigenvalues analytically.
QR Iteration: Computational Expense

A full QR factorization at each iteration costs $O(n^3)$—can we make that cheaper?

Demo: Householder Similarity Transforms [cleared]