Recap: Norms

What’s a norm?

Define *norm*.

- positive definiteness
- semilinearity
- triangle inequality
Norms: Examples

Examples of norms?

\[ \|x\|_\rho = \left( \sum_{i=1}^{\infty} |x_i|^\rho \right)^{1/\rho} \quad \rho \geq 1 \quad \rho = \infty \]

\[ \|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \]

Demo: Vector Norms [cleared]
\[ \leq \|x\| + \|y\| = 2 \]
Does the choice of norm really matter much?

\[ \alpha \| x \| \leq \| x \|^* \leq \beta \| x \| \]

If \( \alpha \) and \( \beta \) exist (and are valid for all \( x \)), then the two norms are called equivalent.
Norms and Errors

If we’re computing a vector result, the error is a vector. That’s not a very useful answer to ‘how big is the error’. What can we do?

\[ d(\hat{x}, \tilde{y}) = \| \hat{x} - \tilde{y} \| \]

Magnitude of error = \( \| \text{true} - \text{approximate} \| \)
Forward/Backward Error

Suppose we want to compute \( y = f(x) \), but approximate \( \hat{y} = \hat{f}(x) \).

What are the forward error and the backward error?

\[
\begin{align*}
\text{(abs.) Forward error} & \quad \Delta y = \pm (y - \hat{y}) \\
\text{(abs.) Backward error} & \quad \Delta x = \pm (x - \hat{x})
\end{align*}
\]
Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$. What’s the (magnitude of) the forward error?

\[
\Delta y = |1.4 - \sqrt{2}| \approx 0.0142
\]

Relative forward error: $\frac{\Delta y}{|y|} \approx 0.01$

(accurate significant digits)

$1.4 - 1.4121 \approx 10^{-5}$
Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$. What’s the (magnitude of) the backward error?

$x$ so that $f(x) = 1.4$

$x \approx 1.96$

**Backward error:**

$$\left| \frac{\Delta x}{x} \right| \approx 0.02$$

**Relative backward error:**

$$\left| \frac{\Delta y}{y} \right| \approx 0.04$$
What do you observe about the relative magnitude of the relative errors?

\[ \text{backward} \geq \text{forward} \]

reality is not always that nice
Sensitivity and Conditioning

What can we say about amplification of error?

$$\left| y - \hat{y} \right| \leq K_{\text{abs}} \left| x - \hat{x} \right|$$

Condition number

$$K_{\text{rel}} = \text{cond}(f) = \max_{x} \frac{|f(x + \Delta x) - f(x)|}{|\Delta x|} \left/ \frac{|f(x)|}{|x|} \right.$$  

$$\text{cond}(g(x)) = \max_{\Delta x} \frac{|g(x + \Delta x) - g(x)|}{|\Delta x|}$$

If $K$ is small:
- good
- well-conditioned

If $K$ is big:
- bad
- poorly conditioned
\[
\frac{|y - y'|}{|y|} \leq \kappa_{\text{vel}} \cdot \frac{|x - \bar{x}|}{|x|}
\]
Example: Condition Number of Evaluating a Function

\[ y = f(x) \]. Assume \( f \) differentiable.

\[
\kappa = \max_{x, \Delta x} \frac{|\Delta y|}{|x| \Delta x} \frac{|y|}{|\Delta x|}
\]

\[
\Delta y = f(x + \Delta x) - f(x) \approx f'(x) \Delta x
\]

\[
\kappa \geq \frac{|\Delta y|/|y|}{|\Delta x|/|x|} \geq \frac{|f'(x)||\Delta x|/|f(x)|}{|\Delta x|/|x|} = \frac{|x \cdot f'(x)|}{|f(x)|}
\]

**Demo:** Conditioning of Evaluating tan [cleared]
\[ f(x + \Delta x) \approx f(x) \]
\[ \approx f(x) + \Delta x \cdot f'(x) \]
Stability and Accuracy

Previously: Considered *problems or questions.*

Next: Considered *methods*, i.e. computational approaches to find solutions.

When is a method *accurate*?

When is a method *stable*?