- HW Length
- Please pay your form posts.
- All due on Oct 12; 30 after class
- Exam 1: book your time.
- HW 3
- HW sol: shown after hw ends

\[ \| A x \| = \max_{\| x \| = 1} \| A x \| \]
Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

▶ \[ \| A \| > 0 \Leftrightarrow A \neq 0. \] **definiteness**

▶ \[ \| \gamma A \| = |\gamma| \| A \| \] for all scalars \( \gamma \). **sublinear**

▶ Obeys triangle inequality \[ \| A + B \| \leq \| A \| + \| B \| \] **\( \Delta \)-ineq.**

But also some more properties that stem from our definition:

\[ \| AB \| \leq \| A \| \| B \| \] **submultiplicativity**

\[ A \Delta x = \Delta b \quad \Rightarrow \quad \Delta x = A^{-1} \Delta b \]
What is the condition number of solving a linear system $Ax = b$?

Input: $b$ with error $\Delta b$

Output: $x$ with error $\Delta x$

$$A(x + \Delta x) = b + \Delta b \Rightarrow A \Delta x = \Delta b$$

Relative error in output

$$\frac{|| \Delta x ||}{|| x ||} = \frac{|| \Delta b ||}{|| b ||} = \frac{|| \Delta x ||}{|| b ||} \frac{|| b ||}{|| x ||}$$

$$\leq \frac{|| A^{-1} \Delta b ||}{|| A x ||} \frac{|| A x ||}{|| x ||}$$

$$\leq || A || || A^{-1} ||$$
Conditioning of Linear Systems: Observations

Showed \( \kappa(\text{Solve } Ax = b) \leq \|A^{-1}\| \|A\| \).

I.e. found an upper bound on the condition number. With a little bit of fiddling, it’s not too hard to find examples that achieve this bound, i.e. that it is sharp.  

So we’ve found the condition number of linear system solving, also called the condition number of the matrix \( A \):

\[
\text{cond}(A) = \kappa(A) = \|A\| \|A^{-1}\| .
\]

\( \checkmark \) (see \$\S\$)

\( \bigodot \) rel. to the (matrix) norm.

\[
\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2
\]
Conditioning of Linear Systems: More properties

- \textit{cond} is relative to a given norm. So, to be precise, use \( \text{cond}_2 \) or \( \text{cond}_\infty \).

- If \( A^{-1} \) does not exist: \( \text{cond}(A) = \infty \) by convention.

What is \( \kappa(A^{-1}) \)?

What is the condition number of matrix-vector multiplication?

\begin{align*}
\text{solve: } A x &= b \\
\text{matvec: } A \vec{x} &= \vec{b} \quad \iff \quad A^{-1} \vec{b} = \vec{x}
\end{align*}

\textbf{Demo:} Condition number visualized [cleared]
\textbf{Demo:} Conditioning of 2x2 Matrices [cleared]
Residual Vector

What is the residual vector of solving the linear system

\[ b = Ax? \]

\[ \hat{r} = \hat{b} - Ax \]

\[ \hat{r} \ y_{\text{computable!}} \]
Residual and Error: Relationship

How do the (norms of the) residual vector $r$ and the error $\Delta x = x - \hat{x}$ relate to one another?

\[
\| \Delta x \| = \| x - \hat{x} \| = \| A^{-1} (b - A\hat{x}) \| = \| A^{-1} r \| \text{ compatible.}
\]

\[
\frac{\| \Delta x \|}{\| r \|} = \frac{\| A^{-1} r \|}{\| r \|} \leq \frac{\| A^{-1} \|}{\| A \|} = \frac{\| A^{-1} \|}{\| A \|} = \text{cond} (A) \frac{\| \Delta x \|}{\| r \|} \leq \text{cond} (A) \frac{\| \Delta x \|}{\| r \|} \leq \text{cond} (A) \frac{\| \Delta x \|}{\| r \|}
\]

"rel. error" \leq \text{cond} (A) \cdot \text{rel. resid}
Changing the Matrix

So far, only discussed changing the RHS, i.e. $Ax = b \rightarrow A\hat{x} = \hat{b}$. The matrix consists of FP numbers, too—it, too, is approximate. I.e.

$$Ax = b \rightarrow A\hat{x} = \hat{b}.$$ 

What can we say about the error due to an approximate matrix?

\[
\Delta x = \hat{x} - x = A^{-1} (A\hat{x} - b) = A^{-1} (A\hat{x} - A\hat{x}) = A^{-1} \Delta A \hat{x}
\]

\[
\|\Delta x\| \leq \|A^{-1}\| \|\Delta A\| \|\hat{x}\| \cdot \frac{\|A\|}{\|A\|} 
\]

\[
\frac{\|\Delta x\|}{\|\hat{x}\|} \leq \text{cond} (A) \cdot \frac{\|\Delta A\|}{\|A\|}
\]

"rel. err."
Changing Condition Numbers

Once we have a matrix $A$ in a linear system $Ax = b$, are we stuck with its condition number? Or could we improve it?

What is this called as a general concept?
In-Class Activity: Matrix Norms and Conditioning

In-class activity: Matrix Norms and Conditioning
Singular Value Decomposition (SVD)

What is the *Singular Value Decomposition* of an $m \times n$ matrix?

\[ A \in \mathbb{R}^{m \times n} \quad A = U \Sigma V^T \]

- col\(U\) = left singular vectors, orthogonal $\in \mathbb{R}^{m \times m}$
- singular values $\Sigma$ = diagonal, has non-negative entries $\in \mathbb{R}^{m \times n}$
  \[ \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \]
- col\(V\) = right singular vectors, orthogonal $\in \mathbb{R}^{n \times n}$

\[ \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \]
Computing the 2-Norm

Using the SVD of $A$, identify the 2-norm.

Let $Q$ be orthogonal: $\|Q \tilde{x}\|_2 = \|\tilde{x}\|_2$

$\|A\|_2 = \|U \Sigma V^T\|_2 = \|\Sigma\|_2 = \sigma_1$

Express the matrix condition number $\text{cond}_2(A)$ in terms of the SVD:

$\text{cond}_2(A) = \frac{\sigma_1}{\sigma_n}$
Not a matrix norm: Frobenius

The 2-norm is very costly to compute. Can we make something simpler?

What about its properties?