Recap: Norms

What's a norm?

Define *norm*.

- positive definiteness
- semilinearity
- triangle inequality

Norms: Examples

$$\left\| \frac{\vec{x}}{\|\vec{x}\|} \right\| = \frac{1}{\|x\|} \cdot \|x\| = 1$$

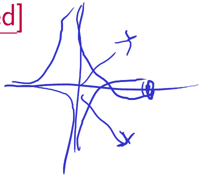
Examples of norms?

$\vec{x} \in \mathbb{R}^n$

$$\|\vec{x}\|_p = \sqrt[p]{\sum_{i=1} |x_i|^p} \qquad p \geq 1 \text{ or } p = \infty$$

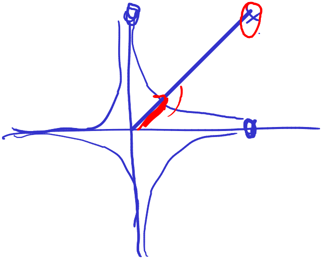
$$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Demo: Vector Norms [cleared]



$|x| + |y| = 1$

$\|x+y\| \leq \|x\| + \|y\|$



$\leq \|x\| + \|y\| = 2$

Norms: Which one?

Does the choice of norm really matter much? $x \in \mathbb{R}^n$

$\|x\|$ $\|x\|^*$

$$\alpha \|x\| \leq \|x\|^* \leq \beta \|x\|$$

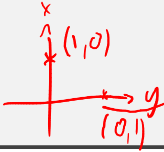
If α, β exist (and are valid for all x), then the two norms are called equivalent.

Norms and Errors

$d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$

If we're computing a vector result, the error is a vector.
That's not a very useful answer to 'how big is the error'.
What can we do?

Magnitude of error $= \|$ true $-$ approximate $\|$ ✓



~~$\|$true$\| - \|$approximate$\|$~~

Forward/Backward Error

Suppose *want* to compute $y = f(x)$, but *approximate* $\hat{y} = \hat{f}(x)$.

What are the forward error and the backward error?

(abs.) Forward error $\Delta y = \pm(\hat{y} - y)$

(abs.) Backward error $\Delta x = \pm(\hat{x} - x)$

$$\begin{tikzcd}
x \arrow[rr, "f"] \arrow[drr, "\hat{f}"] & & y = f(x) \\
\hat{x} \arrow[rr, "f"'] & & \hat{y} = \hat{f}(x)
\end{tikzcd}$$

exists?
unique?

Forward/Backward Error: Example

0.000546
0.000547
2 digits

Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$.
What's the (magnitude of) the forward error?

$$|\Delta y| = 1.4 - 1.4121\ldots \approx 0.0142$$

Relative fwd. error: $\frac{|\Delta y|}{|y|} \approx 0.01$

(accurate significant) digits

1.4
1.4121

10^{-d}

$\to \approx d$ digits

Forward/Backward Error: Example

Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$.
What's the (magnitude of) the backward error?

$f(x) = \sqrt{x}$

$\hat{x}$ so that $f(\hat{x}) = 1.4$

$$\hat{x} \approx 1.96$$

Backward error:

$$|\Delta x| \approx |1.96 - 2| = 0.04$$

Relative backward error

$$\frac{|\Delta x|}{|x|} \approx 0.02$$

Forward/Backward Error: Observations

What do you observe about the relative manitude of the relative errors?

backward ≥ forward

reality is not always that nice

Sensitivity and Conditioning

What can we say about amplification of error?

abs $\to$ $|y-\hat{y}| \le \underbrace{\kappa_{abs}}_{\text{condition number}} \cdot |x-\hat{x}|$

$x: \xrightarrow{f} y$

$\hat{x}: \xrightarrow{f} \hat{y}$

$$\kappa_{rel} = \operatorname{cond}(f) = \max_{x,\Delta x} \frac{|\Delta y| / |y|}{|\Delta x| / |x|}$$

max $\to$ sup

$$\operatorname{cond}(f(x)) = \max_{\Delta x} \frac{\cdots}{\cdots}$$

If κ is small:

- good
- well-conditioned

If κ is big

- bad
- poorly conditioned

$$\frac{|y-\hat{y}|}{|y|} \le \kappa_{rel} \cdot \frac{|x-\hat{x}|}{|x|}$$

Example: Condition Number of Evaluating a Function

$y = f(x)$. Assume f differentiable.

$$\kappa = \max_{x,\Delta x} \frac{|\Delta y| / |y|}{|\Delta x| / |x|}$$

$$\Delta y = f(x+\Delta x) - f(x) \approx f'(x) \cdot \Delta x$$

$$\kappa \geq \frac{|\Delta y| / |y|}{|\Delta x| / |x|} \approx \frac{|f'(x)||\Delta x| / |f(x)|}{|\Delta x| / |x|} = \frac{|x \cdot f'(x)|}{|f(x)|}$$

Demo: Conditioning of Evaluating tan [cleared]

$$f(x+\Delta x) \approx f(x)$$

$$\approx f(x) + \Delta x \cdot f'(x)$$

Stability and Accuracy

Previously: Considered *problems* or *questions*.

Next: Considered *methods*, i.e. computational approaches to find solutions.

When is a method *accurate*?

When is a method *stable*?