What if A doesn't have full runk?
$$S = \{x : ||A\vec{x} - b||_{2} = \min_{x} ||A\vec{y} - b||_{2}\}$$

S = { x : ||Ax-b|| = min || A = -6||2 solution= argmin 11=112

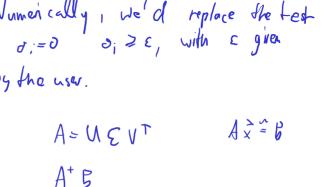
SVD: What's this thing good for? (III)

▶ The minimum norm solution to $Ax \cong b$:

A = USUT

minimizes | | y||2. =) | y||2 = | | v x | 1, = | | x | 1,...

Numerically, we'd replace the test
$$\sigma:=\partial$$
 $\partial_1 \geq \varepsilon$, with ε given by the user.



pseudoinuose: A+ = V E+ UT

Et = ding of for if " or 70"

SVD: Minimum-Norm, Pseudoinverse

What is the minimum 2-norm solution to $Ax \cong b$ and why?



Generalize the pseudoinverse to the case of a rank-deficient matrix.

$$A = u \xi V^T$$

$$A^{\dagger} = v \xi^{\dagger} u^{T}.$$

Comparing the Methods

np. Istsq

Methods to solve least squares with A an $m \times n$ matrix:

hormal eghs:
$$A^{T}A$$
: $mn^{2}/2$

Cholesby: $n^{3}/6$

HouseLolder: $mn^{2}-n^{3}/3$

SVD: $C(mn^{2}+n^{3})$

Demo: Relative cost of matrix factorizations [cleared]

Outline

Introduction to Scientific Computing

Systems of Linear Equations

Linear Least Squares

Eigenvalue Problems
Properties and Transformations
Sensitivity
Computing Eigenvalues
Krylov Space Methods

Nonlinear Equation

Optimizatio

Interpolation

Numerical Integration and Differentiation

Initial Value Problems for ODEs

Boundary Value Problems for ODE

Partial Differential Equations and Sparse Linear Algebra

Fast Fourier Transform

Additional Topics

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

 $ightharpoonup x \neq 0$ is called an *eigenvector* of A if there exists a λ so that

$$A\mathbf{x} = \lambda \mathbf{x}$$
.

- ▶ In that case, λ is called an *eigenvalue*.
- ▶ The set of all eigenvalues $\lambda(A)$ is called the *spectrum*.
- ► The *spectral radius* is the magnitude of the biggest eigenvalue:

$$\rho(A) = \max\{|\lambda| : \lambda(A)\}$$

Finding Eigenvalues

How do you find eigenvalues?

$$A\mathbf{x} = \lambda \mathbf{x} \Leftrightarrow (A - \lambda I)\mathbf{x} = 0$$

$$\Leftrightarrow A - \lambda I \text{ singular} \Leftrightarrow \det(A - \lambda I) = 0$$

 $det(A - \lambda I)$ is called the *characteristic polynomial*, which has degree n, and therefore n (potentially complex) roots.

Does that help algorithmically? Abel-Ruffini theorem: for $n \ge 5$ is no general formula for roots of polynomial. IOW: no.

- For LU and QR, we obtain exact answers (except rounding).
- For eigenvalue problems: not possible—must approximate.

Demo: Rounding in characteristic polynomial using SymPy [cleared]

Multiplicity

What is the *multiplicity* of an eigenvalue?

Actually, there are two notions called multiplicity:

- ► Algebraic Multiplicity: multiplicity of the root of the characteristic polynomial
- Geometric Multiplicity: #of lin. indep. eigenvectors

In general: $AM \geqslant GM$.

If AM > GM, the matrix is called *defective*.

An Example

Give characteristic polynomial, eigenvalues, eigenvectors of

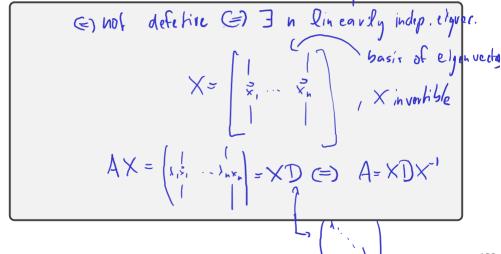
$$(P: (1-\lambda)^{2}$$
Eigenvalues: | , with alg. multiplially 2
$$(| | | (x) = (x)$$

$$(| | (y) = (y)$$
eigenvectors: (x) (=) $\{y = y\}$
yeo.multiplialy: | $\{x,y = x = y\}$ $\{y = 0\}$

Diagonalizability

When is a matrix called diagonalizable?

Acknin C square!



Similar Matrices

Related definition: Two matrices A and B are called similar if there exists an invertible matrix X so that $A = XBX^{-1}$.

In that sense: "Diagonalizable" = "Similar to a diagonal matrix".

Observe: Similar A and B have same eigenvalues. (Why?)

Suppose
$$A\vec{v} = \lambda \vec{v}$$
 $B = \times^{-1}AX$ $\vec{w} = \times^{-1}\vec{v}$
 $B\vec{v} = \times^{-1}A\times\times^{-1}\vec{v} = \times^{-1}A\vec{v} = \times^{-1}\lambda\vec{v}$

Eigenvalue Transformations (I)

What do the following transformations of the eigenvalue problem $A\mathbf{x}=\lambda\mathbf{x}$ do?

Shift. $A \rightarrow A - \sigma I$

$$(A-\sigma T)^{3} = A^{3} - \sigma^{3} = \lambda^{3} - \sigma^{3} = \lambda^{3}$$

Inversion. $A o A^{-1}$

Power $A \rightarrow A^k$

$$A^{\mu} \stackrel{>}{\underset{\sim}{\times}} = \underbrace{A \cdots A}_{\stackrel{\sim}{\underset{\sim}{\times}}} = \lambda^{\mu} \stackrel{\sim}{\underset{\stackrel{\sim}{\underset{\sim}}{\times}}}$$

Eigenvalue Transformations (II)

Polynomial $A \rightarrow aA^2 + bA + cI$

$$\left(\alpha A^{2} + bA + cI\right)_{X=}^{3} \left(\alpha \lambda^{2} + b\lambda + c\right) \gtrsim$$

Similarity $T^{-1}AT$ with T invertible

Sensitivity (I)

Assume A not defective. Suppose $X^{-1}AX = D$. Perturb $A \to A + E$. What happens to the eigenvalues?

1(h I-D) "/ " < 117/

Sensitivity (II)

$$X^{-1}(A+E)X = D+F$$
. Have $\|(\mu I - D)^{-1}\|^{-1} \le \|F\|$.

Demo: Bauer-Fike Eigenvalue Sensitivity Bound [cleared]

Power Iteration

What are the eigenvalues of A^{1000} ?

Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

Further assume $\|\mathbf{x}_i\| = 1$.