- Site looks broken? Shift + click "reload".
- Excel let 2 grades out.

Review:

- Eigenvalue problems: \( AX = \lambda X \) 
- Sensitivity bound: \( A x = \lambda x \) \( (A+E)x = \mu x \)

\[
\max_{\mu} \left| \mu - \text{closest} \lambda_k \right| \leq \text{cond} (X) \cdot \|E\|_2
\]

\( A \) symmetric: \( \Rightarrow \) \( X \) orthogonal \( \Rightarrow \) \( \text{cond}_2 (X) = 1 \)

- Operations
  - \( A^k \Rightarrow \lambda^k \)
  - \( A^{-k} \Rightarrow \frac{1}{\lambda^k} \)
  - \( A^{-1} \Rightarrow \text{eigvec same} \)
  - \( A - \sigma I \Rightarrow \text{eigvec same} \)
Power Iteration

**Demo:** Motivating Power Iteration [cleared]

Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $x_1, \ldots, x_n$. Further assume $\|x_i\| = 1$.

Use random vector: $\tilde{y}_0 = \alpha \tilde{x}_1 + \beta \tilde{x}_2$

$y_{1000} = A^{1000} \tilde{y}_0 = \alpha \lambda_1^{1000} \tilde{x}_1 + \beta \lambda_2^{1000} \tilde{x}_2$

$y_{1000} = A^{1000} \tilde{y}_0 = \alpha \lambda_1^{1000} \tilde{x}_1 + \beta \lambda_2^{1000} \tilde{x}_2$
Power Iteration: Issues?

What could go wrong with Power Iteration?

- $|\lambda_2| = |\lambda_1|$? (includes multiplicity)
- Overflow $\rightarrow$ "normalized power iteration"
- $|\lambda_2| \times |\lambda_1| \Rightarrow$ conv. factor $|\frac{\lambda_2}{\lambda_1}| \approx 1$
  $\Rightarrow$ need lots of iterations
- Only get the first one?
- What if $\alpha = 0$? $\Rightarrow$ actual problem in exact arithmetic?
- $\lambda$ complex?
- $\Rightarrow$ maybe OK w/ rounding?
What about Eigenvalues?

\[ \frac{(A x)_i}{x_i} = \lambda \]

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

\[ A x = \lambda x \Rightarrow \frac{x^T A x}{x^T x} \approx \text{Rayleigh quotient}, \]

\[ A x = \lambda x \Rightarrow \lambda = \lambda \]
Convergence of Power Iteration

What can you say about the convergence of the power method? Say \( \mathbf{v}_1^{(k)} \) is the \( k \)th estimate of the eigenvector \( \mathbf{x}_1 \), and

\[
e_k = \left\| \mathbf{x}_1 - \mathbf{v}_1^{(k)} \right\|.
\]

\[
\mathbf{e}_{n+1} \approx \left| \frac{\lambda_2}{\lambda_1} \right| \mathbf{e}_n
\]

is linearly convergent. (quadratically convergent)

\[
\begin{pmatrix}
\lambda_1 & 1 & 0 \\
0 & \lambda_2 - \sigma & 1 \\
0 & 0 & \lambda_1 - \sigma
\end{pmatrix}
\]
Inverse Iteration

Describe inverse iteration.

\[ x_{u+1} = (A - \sigma I)^{-1} x_u \]

\[ c_{u+1} \approx \begin{vmatrix} \lambda_{\text{closest to } \sigma} - \sigma \\ \lambda_{\text{second closest to } \sigma} - \sigma \end{vmatrix} c_u \]

Complex shifts also OK.
Describe Rayleigh Quotient Iteration.

\[ x_{k+1} = (A - \sigma I)^{-1} x_k \]

\[ \sigma_k = \frac{\sum \frac{A \cdot x_k}{x_k^T x_k}}{\sum x_k^T x_k} \]

Demo: Power Iteration and its Variants [cleared]
In-class activity: Eigenvalues