Power it: (normalized)
\[ \tilde{x}_{k+1} = \tilde{x}_k A \]
\[ x_{k+1} = \tilde{x}_{k+1} / \| \tilde{x}_{k+1} \|_2 \]

Shift -invar:
\[ A \rightarrow (A - \sigma I)^{-1} \]

MQI:
\[ \sigma_k = \frac{x_n^T A x_n}{x_n^T x_n} \]
\[ \alpha < 1 \]
\[ e_{n+1} = \alpha \cdot e_n \]
\[ e_{n+1} \approx \alpha e_{n+1}^2 \]

'Linear' convergence:
'quadratic' conv.
Describe *Rayleigh Quotient Iteration*.

(see above)

**Demo:** Power Iteration and its Variants [cleared]
Schur form

Show: Every matrix is orthonormally similar to an upper triangular matrix, i.e. $A = QUQ^T$. This is called the Schur form or Schur factorization.

Assume $A$ non-defective. Let $\mathbf{v}$ be an eigenvector. Build a basis $i (\mathbf{v}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$ so that $\mathbf{v}^T \mathbf{w}_i = 0$.

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n \end{bmatrix} \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{w}_2^T \\ \vdots \\ \mathbf{w}_n^T \end{bmatrix}$$

$$ \Rightarrow Q_1^T A Q_1 = \mathbf{u}_1$$

$$U = Q_n^T \cdots Q_1^T A Q_1 Q_1 \cdots Q_n$$
A = QUQ^T. For complex λ:
▶ Either complex matrices, or
▶ 2 × 2 blocks on diag.

If we had a Schur form of A, how can we find the eigenvalues?

And the eigenvectors?

\[ \lambda = \text{diag of } U. \]

\[ \begin{align*}
  u - \lambda I &= \begin{pmatrix}
    u_{11} & u_{12} \\
    0 & \lambda
  \end{pmatrix} \\
  \Rightarrow \begin{pmatrix}
    u_{11}^{-1} u_{12} & -1 \\
    0 & \lambda - \lambda
  \end{pmatrix} = 0
\end{align*} \]
All Power Iteration Methods compute one eigenvalue at a time. What if I want all eigenvalues?

- "Deflation": find similarity transform to

\[
\begin{pmatrix}
\lambda & 0 \\
0 & 0
\end{pmatrix}
\]

Rinse, repeat on B.

- Simultaneous iteration...?
Simultaneous Iteration

What happens if we carry out power iteration on multiple vectors simultaneously?

Pick $X_0 \in \mathbb{R}^{n \times p}$ random

$X_{k+1} = AX_k$

(probably needs some normalization)

... silly ... $X_k$ get ill-conditioned

because all columns approach $x_i$, whose $Ax_i = \lambda_{\text{max}} x_i$. 
Orthogonal Iteration

\[
\begin{align*}
\text{rand. } & \quad X_0 \in \mathbb{R}^{n \times p} \quad (p \leq n) \\
\hat{Q}_k \hat{R}_n &= X_k \\
X_{k+1} &= A \hat{Q}_k
\end{align*}
\]
Toward the QR Algorithm

\[ Q_0 R_0 = X_0 \]

\[ x_i = A \hat{Q}_i \]

\[ \hat{Q}_i R_i = X_i = A \hat{Q}_i \Rightarrow \hat{Q}_i^T A \hat{Q}_i = R_i \]

If \( \hat{Q}_n \) converges, \( \hat{Q}_n R_n \hat{Q}_n^T \approx A \)

\[ \hat{X}_n = \hat{Q}_n^T A \hat{Q}_n \approx R_n \]

**Demo:** Orthogonal Iteration [cleared]
QR Iteration/QR Algorithm

\[
\text{Orth. It - } \quad X_0 = A \\
Q_k R_k = X_k \\
X_{k+1} = A Q_k^n \\
\tilde{X}_{k+1} = \bar{R}_k \bar{Q}_n = \bar{Q}_n X_n \bar{Q}_n \Rightarrow X_n \text{ all similar to } A \\
\Rightarrow \text{ have same eigenvals}
\]

Claim: \( Q_k \) in Orth. It can be chosen to be \( Q_k = \bar{Q}_n Q_k \bar{Q}_n Q_k \)

\[
\tilde{X}_k = X_{k+1}
\]
Proof sketch: Equivalence of QR iteration/Orth. iteration

**Orthogonal Iteration (no bars)**

- $X_0 := A$
  - $Q_0 R_0 := X_0$, where we may choose $Q_0 = \bar{Q}_0$
  - $\hat{X}_0 = Q_0^H A Q_0 = Q_0^H Q_0 R_0 Q_0 = R_0 Q_0$
- $X_1 := A Q_0$
  - $Q_1 R_1 := X_1$, and because of $X_1 = Q_0 Q_0^H A Q_0 = Q_0 \bar{Q}_1 \bar{R}_1$
  - we may choose $Q_1 = Q_0 \bar{Q}_1 = \bar{Q}_0 \bar{Q}_1$.
- $\vdots$

**QR Iteration (with bars)**

- $\bar{X}_0 := A$
  - $\bar{Q}_0 \bar{R}_0 := A$
  - $\bar{X}_1 := \bar{R}_0 \bar{Q}_0 = \hat{X}_0$
  - $\bar{Q}_1 \bar{R}_1 := \bar{X}_1$
- $\bar{X}_2 := \bar{R}_1 \bar{Q}_1$
  - $\bar{X}_2 = Q_1^H A Q_1 = \hat{X}_1$
- $\vdots$
QR Iteration: Forward \textit{and} Inverse

QR iteration may be viewed as performing \textit{inverse iteration}. How?