- Examled 2
- HW8
Solving Nonlinear Equations

What is the goal here?

\[ f(x) = 0 \]

\( \subset 10 \) cases:  \( f(x) = 0 \)
Showing Existence

How can we show existence of a root?

- Intermediate value theorem: if \( f \) is continuous,

- Inverse function theorem: \( f \) is invertible at some point \( x \in \mathbb{R}^n \)

  \[ \frac{\partial}{\partial x} f(x) = \text{invertible} \]

  \[ \Rightarrow \text{there exists } \epsilon > 0 \text{ so that } f \text{ is invertible in } B(x, \epsilon) \]

- Contraction mapping theorem

  A function \( g : \mathbb{R}^n \to \mathbb{R}^n \) is called contractive if there exists \( 0 < \rho < 1 \) so that

  \[ \| g(x) - g(y) \| \leq \rho \| x - y \| \]

  On a closed set \( S \subseteq \mathbb{R}^n \) with \( g(S) \subseteq S \) then exists a fixed point \( x^* : g(x^*) = x^* \).
What is the sensitivity/conditioning of root finding?

\[
\text{cond (root finding)} = \text{cond} \left( \frac{\partial}{\partial \mathbf{x}} \right)
\]

abs

What are multiple roots?

Example: \( f(x) = (x - x_0)^2 \cdot \Phi(x) \)

Def: \( \Phi(x^*) = 0, \Phi'(x^*) = 0 \)

How do multiple roots interact with conditioning?

Inverse is steep, therefore conditioning is poor.
Rates of Convergence

What is *linear convergence*? *quadratic convergence*?

\[ \hat{E}_n = \hat{u}_n - \hat{u} \]

An iterative method converges with rate \( r \) iff

\[ \lim_{k \to \infty} \frac{\|\hat{E}_{n+1}\|}{\|\hat{E}_n\|} = C > 0 \]

- \( r = 1 \): Linear conv
- \( r = 2 \): Quadratic
- \( r > 1 \): Superlinear

Previous: cardinal def: \( \|E_{n+1}\| \leq C \cdot \|E_n\| \)

\( \|E_{n+1}\| \leq C \cdot \|E_n\| \)
Demo: Rates of Convergence [cleared]

Characterize linear, quadratic convergence in terms of the ‘number of accurate digits’.

Linear: gains fixed number of digits per it.

Quadratic: doubles number of digits.
Stopping Criteria

Comment on the ‘foolproof-ness’ of these stopping criteria:

1. \(|f(x)| < \varepsilon\) (‘residual is small’)
2. \(|x_{k+1} - x_k| < \varepsilon\)
3. \(|x_{k+1} - x_k| / \|x_k\| < \varepsilon\)

\[\begin{align*}
\text{Long ways from the root. } &\Rightarrow \text{ Fail} \\
\text{Method gets stuck } &\Rightarrow \text{ Fail} \\
\text{Goes bad if } \|x_{k+1}\| \text{ is small} &\Rightarrow \text{ Fail}
\end{align*}\]
Demo: Bisection Method [cleared]

What’s the rate of convergence? What’s the constant?

Linear
Fixed Point Iteration

\[ x_0 = \text{(starting guess)} \]

\[ x_{k+1} = g(x_k) \]

**Demo:** Fixed point iteration [cleared]

When does fixed point iteration converge? Assume \( g \) is smooth.

Let \( x^* \) be the fixed pt. with \( g(x^*) = x^* \).

If \( |g'(x^*)| < 1 \), then there exists a neighborhood where we have convergence.

\[ e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*) \]
Error in FPI: $e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$

What if $g'(x^*) = 0$?

Does not imply that $g'(\Theta_n) = 0$, i.e. can.

$g'(x^* + h) \approx C|h|$

$|h| \approx |e_n|$

$\Rightarrow \|e_{n+1}\| \approx C \cdot h \cdot |e_n| \approx C \cdot \|e_n\|^2$
Newton’s Method

Derive Newton’s method.

\[ p(x_n + h) \approx p(x_n) + p'(x_n) h + O(h^2) \]

Solve

\[ p(x_n) + p'(x_n) h = 0 \quad \text{for} \ h. \]

Use

\[ x_{n+1} = x_n + h. \]

\[ x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} = g(x_n) \]

Demo: Newton’s method [cleared]
Convergence and Properties of Newton

What’s the rate of convergence of Newton’s method?

Drawbacks of Newton?

Demo: Convergence of Newton’s Method [cleared]