• HW 14 (cc, due tomorrow)
• 4CH 2 (+5% if in by tomorrow, otherwise during finals)
• Final (Dec 8-12)
• ICES (please fill them out!)
Euler’s Method

Discretize the IVP

\[
\begin{cases}
  y'(t) = f(y) \\
  y(t_0) = y_0
\end{cases}
\]

- Discrete times: \( t_1, t_2, \ldots, \) with \( t_{i+1} = t_i + h \)
- Discrete function values: \( y_k \approx y(t_k) \).

\[
y(t) = y_0 + \int_{t_0}^{t} f(y_c) \, dc
\]
Euler’s method: Forward and Backward

\[ y(t) = y_0 + \int_{t_0}^{t} f(y(\tau)) d\tau, \]

Use ‘left rectangle rule’ on integral:

\[ \hat{y}_{k+1} = \hat{y}_k + h \cdot f(\hat{y}_k) \quad \text{FE Euler} \]

Use ‘right rectangle rule’ on integral:

solve for \( y_{k+1} \)

\[ \hat{y}_{k+1} = \hat{y}_k + h \cdot f(\hat{y}_k) \quad \text{BE Euler} \]

**Demo:** Forward Euler stability [cleared]
Let \( u_k(t) \) be the function that solves the ODE with the initial condition \( u_k(t_k) = y_k \). Define the local error at step \( k \) as...

\[
\ell_k = y_k - u_{k-1}(t_k)
\]

Define the global error at step \( k \) as...

\[
g_k = y(t_k) - y_n
\]
**About Local and Global Error**

Is global error $= \sum$ local errors?

No, just like compound interest

A time integrator is said to be *accurate of order $p$* if...

$x_n = O\left(h^{p+1}\right)$
A time integrator is said to be *accurate of order* $p$ if $\ell_k = O(h^{p+1})$

This requirement is one order higher than one might expect–why?

$$\frac{1}{h} \cdot O(h^{p+1}) = O(h^p)$$

for sum of local errors

# steps

$\to$ get to time $O(1)$
Stability of a Method

Find out when forward Euler is stable when applied to \( y'(t) = \lambda y(t) \).

\[
y_{k+1} = y_n + h \lambda y_n \\
= y_n (1 + h \lambda) \\
= (1 + h \lambda)^{k+1} y_0
\]

\[
\text{stable \iff} \quad |1 + h \lambda| \leq 1
\]

stability region for

\( \text{fw Euler} \)
Stability: Systems

What about stability for systems, i.e. \( y'(t) = Ay(t) \)?

- Diagonalize the system

\[
\begin{align*}
A &= VDV^T \\
D &= VAV^{-1} \\
\omega_i &= \lambda_i \Rightarrow \omega_i = \lambda_i \Rightarrow w_i = \lambda_i \omega_i \\
D' = D \\
w' &= Dw \\

w_{n+1} &= V y_{n+1} = V (y_n + Ay_n) \\
&= V y_n + VAV^{-1}DVy_n \\
&= w_n + VV^T D V y_n
\end{align*}
\]
Stability: Nonlinear ODEs

What about stability for nonlinear systems, i.e.

\[ y'(t) = f(y(t)) \]?

\[ e(t) = y(t) - \hat{y}(t) \]

\[ e'(t) = f(y(t)) - f(\hat{y}(t)) \leq \int f'(y) e(t) \] (1)

\[ \dot{e}' = \int f'^2 \]

\[ \text{+ HOT vs. stab. region} \]

→ look at eigenvalues of Jacobian of \( f \)
Stability for Backward Euler

Find out when backward Euler is stable when applied to $y'(t) = \lambda y(t)$.

\[ y_{n+1} = y_n + h \lambda y_{n+1} \]
\[ y_{n+1} \left( 1-h\lambda \right) = y_n \]
\[ y_{n+1} = \frac{1}{1-h\lambda} \quad y_n = \left( \frac{1}{1-h\lambda} \right)^n y_0 \]

Stable $\iff \left| \frac{1}{1-h\lambda} \right| \leq 1$

**Demo:** Backward Euler stability [cleared]
Stiff ODEs: Demo

Demo: Stiffness [cleared]

\[
\lambda = \frac{\text{Re}(\lambda)}{\text{Im}(\lambda)} > 0 \quad \rightarrow \text{ODE unstable}
\]

\[
\text{Re}(\lambda) < 0 \quad \text{(ODE asymptotically stable)}
\]

\[
\Rightarrow \text{pick any time step, it will be stable anyway}
\]

If a method is stable for all \( h \) when \( \text{Re}(\lambda) < 0 \),

"\( A \)-stable"
\[ y' = -100y + 100t + 101 \]

\[ y_{n+1} = y_n + h \left(-100y_{n+1} + 100t + 101\right) \]

\[ (1 + 100h) \cdot y_{n+1} = y_n + h (100t + 101) \]
‘Stiff’ ODEs

- Stiff problems have *multiple time scales*.  
  (In the example above: Fast decay, slow evolution.)
- In the case of a stable ODE system

\[
y'(t) = f(y(t)),
\]

stiffness can arise if \( J_f \) has eigenvalues of very different magnitude.
Why not just ‘small’ or ‘large’ magnitude?

What is the problem with applying explicit methods to stiff problems?
Predictor-Corrector Methods

Idea: Obtain intermediate result, improve it (with same or different method).

For example:

- "Predictor": \( \tilde{y}_{n+1} = y_n + hf(y_n) \)
- "Corrector": \( y_{n+1} = y_n + \frac{h}{2} \left( f(y_n) + f(\tilde{y}_{n}) \right) \)

Heun's method

\( \rightarrow \) 2nd order global accuracy
Runge-Kutta/‘Single-step’/‘Multi-Stage’ Methods

Idea: Compute intermediate ‘stage values’, compute new state from those:

\[
\begin{align*}
    r_1 &= D(t_n + c_1 h, y_n + h (a_{11} r_1 + \cdots + a_{1s} r_s)) \\
    r_s &= D(t_n + c_s h, y_n + h (a_{s1} r_1 + \cdots + a_{ss} r_s)) \\
    y_{n+1} &= y_n + h (b_1 r_1 + \cdots + b_s r_s)
\end{align*}
\]

Can summarize in a Butcher tableau:

```
  \begin{array}{c|ccc}
    c_1 & a_{11} & \cdots & a_{1s} \\
    \downarrow & \uparrow & \cdots & \uparrow \\
    c_s & a_{s1} & \cdots \\
    \downarrow & \uparrow & \cdots & \uparrow \\
    y_{n+1} & b_1 & \cdots & b_s \\
  \end{array}
```
Runge-Kutta: Properties

When is an RK method explicit?

nonzeros only below the diagonal

When is it implicit?

otherwise

When is it diagonally implicit? (And what does that mean?)

nonzeros not above the diagonal

\rightarrow can solve one at a time.
Stuff Heun’s method into a Butcher tableau:

1. $\tilde{y}_{k+1} = y_k + hf(y_k)$
2. $y_{k+1} = y_k + \frac{h}{2}(f(y_k) + f(\tilde{y}_{k+1}))$. 

\[
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & & \\
\end{array}
\]
RK4

What is RK4?

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\[ \frac{1}{6} \quad \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{6} \]

\[ \frac{4}{2} \]

**Demo:** Dissipation in Runge-Kutta Methods [cleared]
Why does the idea of stability regions still apply to more complex time integrators (e.g. RK?)

**Demo:** Stability regions [cleared]
More Advanced Methods

Discuss:

- What is a good cost metric for time integrators?
- AB3 vs RK4
- Runge-Kutta-Chebyshev
- LSERK and AB34
- IMEX and multi-rate
- Parallel-in-time ("Parareal")

![Diagram with plots for different methods: ab3, ab34, lserk, rk4.](image-url)