- hw 13
- 4C1 assignment
- exam let 5 page grades

\[ f(x) - p_{n-1}(x) = \sum_{i=0}^{n} \frac{f^{(i)}(\xi)}{i!} (x-x_1)(x-x_2)\cdots(x-x_n) \]
Error Result: Simplified Form

Boil the error result down to a simpler form.

\[ \left| f^{(n)}(x) \right| \leq M \quad (x \in I) \]

interval length \( |I| = h = b-a \)

\( (x-x_i) : \) if \( x \in I \), \( |x-x_i| \leq \frac{1}{2} \)

\[ \text{Error}(h) = \left| f(x) - p_{n-1}(x) \right| \leq CM \cdot h^n \]

\[ \text{Error} = O(h^n) \quad (\text{as } h \to 0) \]

\( \in C^{n-\text{th}} \text{ order convergence} \)

- **Demo:** Interpolation Error [cleared]
- **Demo:** Jump with Chebyshev Nodes [cleared]
Going piecewise: Simplest Case

Construct a piecewise linear interpolant at four points.

\[
\begin{align*}
& f_1 = a_1 x + b_1 \\
& f_2 = a_2 x + b_2 \\
& f_3 = a_3 x + b_3 \\
& f_1(x_0) = y_0 \\
& f_1(x_1) = y_1 \\
& f_2(x_1) = y_1 \\
& f_2(x_2) = y_2 \\
& f_3(x_2) = y_2 \\
& f_3(x_3) = y_3 \\
\end{align*}
\]

Why three intervals?

Middle
Piecewise Cubic (‘Splines’)

\[ f_1(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1 \]
\[ f_2(x) = a_2 x^3 + b_2 x^2 + c_2 x + d_2 \]
\[ f_3(x) = a_3 x^3 + b_3 x^2 + c_3 x + d_3 \]

\[ f_1(x_0) = y_0 \]
\[ f_1(x_1) = y_1 \]
\[ f_2(x_1) = y_1 \]
\[ f_2(x_2) = y_2 \]
\[ f_3(x_2) = y_1 \]
\[ f_3(x_3) = y_3 \]

\[ f_1'(x_0) = f_2'(x_1) \]
\[ f_2'(x_1) = f_3'(x_2) \]
\[ f_1''(x_0) = f_2''(x_1) \]
\[ f_2''(x_1) = f_3''(x_2) \]

\[ f_1''(x_0) = 0 \]
\[ f_2''(x_1) = 0 \]
\[ f_3''(x_3) = 0 \]
### Piecewise Cubic (‘Splines’): Accounting

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<thead>
<tr>
<th>$x_0, y_0$</th>
<th>$x_1, y_1$</th>
<th>$x_2, y_2$</th>
<th>$x_3, y_3$</th>
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<tr>
<td>$f_1$</td>
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<tr>
<td>$a_1 x^3 + b_1 x^2 + c_1 x + d_1$</td>
<td>$a_2 x^3 + b_2 x^2 + c_2 x + d_2$</td>
<td>$a_3 x^3 + b_3 x^2 + c_3 x + d_3$</td>
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Outline

Introduction to Scientific Computing

Systems of Linear Equations

Linear Least Squares

Eigenvalue Problems

Nonlinear Equations

Optimization

Interpolation

Numerical Integration and Differentiation

Numerical Integration
Quadrature Methods
Accuracy and Stability
Gaussian Quadrature
Composite Quadrature
Numerical Differentiation
Richardson Extrapolation

Initial Value Problems for ODEs

Boundary Value Problems for ODEs

Partial Differential Equations and Sparse Linear Algebra

Fast Fourier Transform

Additional Topics

\[ f(x) \approx p_{n-1}(x) = \sum \alpha_i \phi_i(x) \]

\[ f'(x) \approx p_{n-1}'(x) = \sum \alpha_i \phi_i'(x) \]

\[ \int_a^b f(x) \, dx \approx \int_a^b p_{n-1}(x) \, dx = \sum \alpha_i \int_a^b \phi_i(x) \, dx \]
Numerical Integration: About the Problem

What is numerical integration? (Or quadrature?)

Given \( a, b, f \), approximate: \( \int_a^b f(x) \, dx \)

What about existence and uniqueness?

- \( f \) integrable (Riemann/Lebesgue)
- \( f \) is unique for pw. continuous/bounded
Derive the (absolute) condition number for numerical integration.

\[
\int_a^b f(x) \, dx = \int_{a'}^{b'} e(x) \, dx
\]

\[
\left| \int_a^b f(x) \, dx - \int_{a'}^{b'} f(x) \, dx \right| \\
< \int_a^b |e(x)| \, dx \leq (b-a) \max_{x \in (a,b)} |e(x)|
\]
Interpolatory Quadrature

Design a quadrature method based on interpolation.

\[
\int_a^b f(x) \, dx \approx \sum_{i=1}^n \alpha_i \varphi_i(x)
\]

\[
\int_a^b f(x) \, dx = \sum_{i=1}^n \alpha_i \int_a^b \varphi_i(x) \, dx
\]

**Goal:** \( \int_a^b f(x) \, dx = \sum_{i=1}^n \omega_i \varphi(x_i) \)
Let $l_i$ be the Lagrange polynomials for nodes $\{x_i\}_{i=1}^n$.

$$\int_a^b \phi(x) dx \approx \sum_{i=1}^n g(x_i) \sum_{i=1}^n l_i(x) w_i$$

Nodes: weights.

Equispaced: Newton-Cotes quadrature

Chebyshev nodes: Clenshaw-Curtis quadrature (+ poly basis)
Interpolatory Quadrature: Computing Weights

How do the weights in interpolatory quadrature get computed?

**Demo:** Newton-Cotes weight finder [cleared]
Midpoint rule:

Turns out: It's general: quadrature with odd numbers of points have one "bonus" degree of freedom
Examples and Exactness

To what polynomial degree are the following rules exact?

Midpoint rule \((b - a)f\left(\frac{a+b}{2}\right)\)

Trapezoidal rule \(\frac{b-a}{2}(f(a) + f(b))\)

Simpson’s rule \(\frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\)
Interpolatory Quadrature: Accuracy

Let $p_{n-1}$ be an interpolant of $f$ at nodes $x_1, \ldots, x_n$ (of degree $n - 1$)

Recall

$$\sum_i \omega_i f(x_i) = \int_a^b p_{n-1}(x) \, dx.$$ 

What can you say about the accuracy of the method?

\[
\begin{align*}
|\int_a^b p(x) \, dx - \int_a^b p_{n-1}(x) \, dx| &
\leq \int_a^b |p(x) - p_{n-1}(x)| \, dx \\
&\leq (b-a) \| p - p_{n-1} \|_\infty \\
&\leq (b-a) C h^n \| f^{(n)} \|_\infty \\
&= C h^{n+1} \| f^{(n)} \|_\infty
\end{align*}
\]
## Quadrature: Overview of Rules

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<td>S. 3/8</td>
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<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
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</tbody>
</table>

- $n$: number of points
- “Deg.”: Degree of polynomial used in interpolation ($= n - 1$)
- “Ex.Int.Deg.”: Polynomials of up to (and including) this degree actually get integrated exactly. (including the odd-order bump)
- “Intp.Ord.”: Order of Accuracy of Interpolation: $O(h^n)$
- “Quad.Ord. (regular)”: Order of accuracy for quadrature predicted by the error result above: $O(h^{n+1})$
- “Quad.Ord. (w/odd)”: Actual order of accuracy for quadrature given ‘bonus’ degrees for rules with odd point count

**Observation:** Quadrature gets (at least) ‘one order higher’ than interpolation—even more for odd-order rules. (i.e. more accurate)
Interpolatory Quadrature: Stability

Let $p_n$ be an interpolant of $f$ at nodes $x_1, \ldots, x_n$ (of degree $n - 1$)

Recall

$$\sum_i \omega_i f(x_i) = \int_a^b p_n(x) dx$$

What can you say about the stability of this method?

So, what quadrature weights make for bad stability bounds?